Lecture 10 — 10/11

- Simple question: why $\vec{L}^2 |\ell\rangle = \hbar^2 \ell(\ell+1) |\ell\rangle$? → math!
  (same for $[\vec{L}^2, L_z] = 0$ and $L_z |m\rangle = m |m\rangle$)
- How do I figure whether I am in the interaction picture or the rotating frame? When to use which? What about the RWA?

↳ Let' summarize: $\boxed{\text{Interaction picture}}$

① $H = H_0 + V \Rightarrow H_{int} = e^{-i\frac{H_0}{\hbar}t} V e^{i\frac{H_0}{\hbar}t}$; $|\psi_I\rangle = e^{-i\frac{H_0}{\hbar}t} |\psi_s\rangle$

(in our cases: $e^{-i\frac{H_0}{\hbar}t} = e^{-i\omega_0 t}$ typically)

② With $V \propto e^{\pm i\omega t} \Rightarrow$ get terms $e^{\pm i(\omega \pm \omega_0)t}$

$\boxed{\text{RWA}}$: neglect "fast rotating", "counter-rot" terms
(when is this a good approx? Bloch-Siegert shift small.)

③ Define $\Delta \equiv \omega - \omega_0 \Rightarrow \exists$ time-dep terms w/ $e^{\pm i\Delta t}$

How does H_I now look? Typically: $\begin{pmatrix} 0 & \alpha e^{i\Delta t} \\ \alpha^* e^{-i\Delta t} & 0 \end{pmatrix}$

④ $\boxed{\text{Rotating frame}}$

Hamiltonian above leads to explicitly time dependent eom s, with a potentially fast oscillation $e^{\pm i\Delta t} \Rightarrow$ go to rotating frame by incorporating this factor into variable,

e.g. $c_e \to \tilde{c}_e = c_e e^{\pm i\Delta t}$

or $\rho_{eg} \to \tilde{\rho}_{eg} = \rho_{eg} e^{\pm i\Delta t}$

This leads to $\dot{c}_e = (\dot{\tilde{c}}_e \mp i\Delta \tilde{c}_e) e^{\mp i\Delta t}$

$\dot{\rho}_{eg} = (\dot{\tilde{\rho}}_{eg} \mp i\Delta \rho_{eg}) e^{\mp i\Delta t}$

Typically, these terms cancel out.

Plugging back into $H_I \to \tilde{H}_I = \begin{pmatrix} -\Delta & \alpha \\ \alpha^* & 0 \end{pmatrix} = e^{\mp i\Delta t} H_I e^{\pm i\Delta t}$

→ for atom number Z (but only one e^-)

$$E_Z = -Z^2 \frac{Ry}{n^2}$$

$$R_{nl}(\rho) \propto \rho^{n+1} e^{-\rho/2} L_{n-l-1}(\rho)$$

↓ "Laguerre polynomials"

normalized: $\int_0^\infty |R(r)|^2 r^2 \, dr = 1$

Degeneracies: (given E/n)

① l, has n values $(0, \dots, n-1)$

② m, has $2l+1$ values $(-l, \dots, +l)$

(③ spin: 2 values)

each n has $2 \sum_{l=0}^{n-1} (2l+1) = 2n^2$ degenerate wavefcts.

②, ③ lifted by "relativistic effects" (Dirac)

① lifted only by vacuum fluctuations: Lamb shift

E > 0 (continuum states)

no limit on b

- general properties

(i) fixed l, small r ⇒ wave fct independent except for general scaling $n^{-3/2}$

(ii) fixed n: changing l affects only short range (for $r \to \infty$: effect of potential barrier negligible)

(iii) size: $\langle r \rangle = \frac{1}{2}\left(3n^2 - \ell(\ell+1)\right) a$

(iv) $\Delta E = Z^2 Ry \left(\frac{1}{n^2} - \frac{1}{n'^2}\right)$

c) Angular momentum

(What happens for $2e^-$? or for combination of orbital ang. mom. and spin?)

in general (any ang. mom): $\vec{J} = \vec{r} \times \vec{p}$

$$\vec{J} \times \vec{J} = i\hbar \vec{J} \quad (*)$$

$$\left.\begin{aligned} \vec{J}^2 |j, m_j\rangle &= \hbar^2 j(j+1) |j, m_j\rangle \\ \hat{J}_z |j, m_j\rangle &= \hbar m_j |j, m_j\rangle \end{aligned}\right\} (**)$$

$(*)$, $(**)$ are equivalent

$$\Rightarrow \hat{J}_\pm |j, m_j\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}\, |j, m_j \pm 1\rangle$$

<u>Addition:</u> $\vec{J} = \vec{J}_1 + \vec{J}_2$

$$[\vec{J}^2, \vec{J}_{1/2}^2] = [\vec{J}, \vec{J}_{1/2}^2] = 0$$

$$[\vec{J}^2, \vec{J}_{1/2}] \neq 0$$

(because: $\vec{J}^2 = \vec{J}_1^2 + \vec{J}_2^2 + 2\vec{J}_1 \cdot \vec{J}_2$)

- <u>Eigenstates (of coupled</u> ang. mom. operators)

1) "uncoupled representation":

$|j_1, m_1, j_2, m_2\rangle$: good q. numbers (full desc. of system)

but <u>not</u> eigenstate of $\vec{J}^2$

2) "coupled representation"

$|j, m, j_1, j_2\rangle$ (also good q. nb.)

eigenstate of $\vec{J}^2$, J_z, $\vec{J}_1^2$, $\vec{J}_2^2$, but not of $\hat{J}_{1z}$, $\hat{J}_{2z}$