Physics 285a Problem Set 1

posted September 21, 2023, due September 26, 2023

Problem 1. Dimensional Analysis:

(a) Assume you know nothing about general relativity, but you know about gravity and special relativity. Using dimensional analysis, what would be your guess what the basic length constant is for an object of mass M? How does this relate to the real Schwarzschild radius? (Use a similar method as we used in class.)

(b) Make up your own dimensional analysis question, relating to AMO, or a classical E&M equivalent. Feel free to use ChatGPT for this part, but if you do: (i) Tell us so; (ii) make sure that this is not nonsense. (I tried and nonsense ensued....)

Problem 2. Coherent states: Energy eigenstates of a mechanical oscillator or a radiation mode field oscillator do not oscillate. They are boring "stationary" states. Coherent states are quantum states that correspond most closely to classical oscillations. Interestingly, a laser produces a coherent state of light, which is sometimes loosely referred to as "classical light."

This problem might seem to be quite a bit beyond of what was done in class so far. We provide step-by-step instructions, so give it a try.

Let ω be the resonant angular frequency of the oscillator. We typically refer to the energy eigenstates of the oscillator as number states $|n\rangle$ since they describe the number of quanta in the radiation field (i.e. the "number of photons") in the field in this case.

1. **Classical oscillator.** Define a complex number α in terms of the displacement of the oscillator from equilibrium, z, and the oscillator momentum, p, using

$$\sqrt{\hbar\omega}\,\alpha \;\equiv\; \sqrt{\frac{m\omega^2}{2}}z + i\frac{p}{\sqrt{2m}}. \tag{1}$$

   (a) Describe the oscillator motion using a first order differential equation for α, and solve the equation.

   (b) What is the total energy of the oscillator in terms of α?

2. **Quantum oscillator.**

   (a) Define a lowering operator a and a raising operator $a^\dagger$ using the typical convention. Derive an equation motion for the average value of the lowering operator $\langle a\rangle$, and solve the equation.

   Feel free to have ChatGPT help with this – but tell us and make sure that the solution makes sense!

   (b) What is the total energy in terms of the average value $\langle a^\dagger a\rangle$?

3. **Correspondence.** Note the correspondence of the classical and quantum equation of motion, and between the classical and quantum expressions for the energy – a correspondence which becomes better for larger excitations.

(a) Show that quantum states that oscillate like a classical oscillator have $|\alpha|^2 = \langle a^\dagger a\rangle$ and $\alpha = \langle a\rangle$, and show that both of these conditions are met for coherent states $|\alpha\rangle$ defined by $a|\alpha\rangle = \alpha|\alpha\rangle$.

(b) Solve this equation using an expansion for the coherent states in terms of the complete set of number states. Express a normalized coherent state as a superposition of number states.

This is pretty straightforward – but feel free to use ChatGPT to get started on this question. Be sure to double-check the output!

4. **Properties of coherent states.** Assume you are in a coherent state $|\alpha\rangle$.

(a) Let $\overline{n} = \langle N\rangle$ be the average of the number operator $N = a^\dagger a$, and compute $\overline{n}$ for a coherent state. For a radiation field, this is the average number of quanta in the field. For a mechanical oscillator, this is the average quantum number/excitation.

(b) Show that the probability P_n to be in a number state $|n\rangle$ is a Poisson distribution which you can write in terms of n and $\overline{n}$.

(c) What is the approximate maximum of the distribution for large $\overline{n}$? Hint: use Stirling's approximation $\log n! \approx n \log n - n$, and the fact that $\log f(x)$ and $f(x)$ are maximum at the same x.

(d) What is the approximate width of the distribution? Use the rms deviation of N, defined by $\Delta N = \sqrt{\langle N^2\rangle - \langle N\rangle^2}$, to prove a quantitative answer in terms of $\overline{n}$.

5. **Uncertainties.**

(a) Calculate the rms uncertainties in z and in p for a coherent state $|\alpha\rangle$. These are defined as $\Delta z = \sqrt{\langle z^2\rangle - \langle z\rangle^2}$ and $\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2}$.

(b) Calculate the rms uncertainties in z and in p for the three lowest energy eigenstates of the oscillator.

(c) Sketch the rms uncertainties in the scaled variables $\tilde{z} = z\sqrt{m\omega/(2\hbar)}$ and $\tilde{p} = p/\sqrt{2m\hbar\omega}$ for each of the four cases calculated above. Superimpose a curve which represents the limit provided by the the uncertainty principle.

Problem 3. Anharmonic Oscillator: To establish the "language of resonance" we will quickly review the damped harmonic oscillator. In reality, however, the many oscillators encountered in physics (e.g. molecular vibrations we will consider later) tend to be anharmonic. Here we explore anharmonic oscillators with a simple example. Note that x, y, and z are spatial coordinates that are all real.

(a) Add to the oscillator described by $\ddot{z} + \gamma\dot{z} + \omega_o^2 z = f\cos(\omega t)$ a small restoring force term az^3. What is the potential energy for the anharmonic oscillator? How does the steady-state oscillator energy depend upon the drive frequency? (Hints: 1. Recall the solution for $a = 0$ and solve this perturbatively, taking $z \to z_0 + az_1$, where z_0 is the solution for $a = 0$. 2. To calculate the energy in the steady state, average over one period. 3. You may wish to use the identity $\cos^3 x = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x$. This will result in two driving terms with different frequencies. One can solve this exactly using the method of undetermined coefficients (or Mathematica), but it turns out that the higher order frequency term will not contribute to the energy to first order in a (when averaged over one cycle) and can be dropped.)

(b) For a perfectly harmonic Penning trap the electric potential is

$$V = \frac{V_o}{2} \frac{z^2 - \frac{1}{2}x^2 - \frac{1}{2}y^2}{d^2}$$

Derive the potential of the Penning Trap by solving for an arbitrary potential energy for $r = (x^2 + y^2 + z^2)^{1/2} \ll d$ and with reflection symmetry across the xy plane by using the solution to Laplace's equation in terms of spherical coordinates up to order 2. You may assume axial symmetry to simplify the arbitrary potential. Show that the arbitrary potential differs from the Penning Trap for higher orders.

(c) Identify where the harmonic potential and the anharmonic potential arose (in part a) in the above expansion. When would we want to make an anharmonic potential?

(d) Obtain the result of part (a), namely the dependence of the oscillator frequency on the oscillation energy, as a limit of a quantum calculation of the anharmonicity. To do so, write down the Hamiltonian for the anharmonic oscillator and compute the energy correction using perturbation theory.