Lecture 3 - 9/13

(Q $\longrightarrow$ "splitting the line"
exp's & defs.)

Two - Level - Systems

avoided crossing
coupling
transitions
rotating frame
rotating wave approx.

Note: "Spectroscopy" is typically: set drive on a frequency ω, measure response by system - e.g. fluorescence, absorption strength, etc. Strength of response allows to draw [peak] and thus, to determine ω_0.

Note: "splitting the line" —

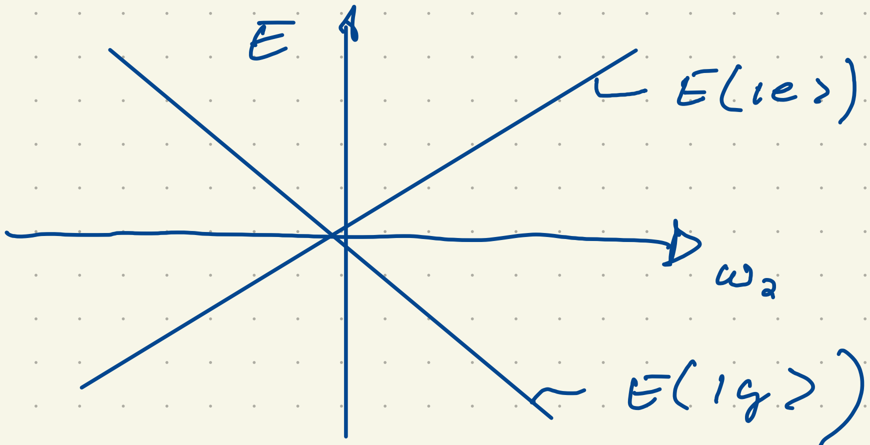
c) Coupling & level crossing

Two states: $|e\rangle$, $|g\rangle$

Uncoupled:

$$\left.\begin{aligned} H|e\rangle &= \tfrac{1}{2}\hbar\omega_0 |e\rangle \\ H|g\rangle &= -\tfrac{1}{2}\hbar\omega_0 |g\rangle \end{aligned}\right\} \begin{aligned} &= \tfrac{1}{2}\hbar\omega_0 \begin{pmatrix}1\\0\end{pmatrix} \\ &= -\tfrac{1}{2}\hbar\omega \begin{pmatrix}0\\1\end{pmatrix} \end{aligned}$$

$$H \to \tfrac{1}{2}\hbar \begin{pmatrix}\omega_0 & 0 \\ 0 & -\omega_0\end{pmatrix}$$



level crossing @ $\omega_0 = 0$

Add coupling:

$$H = \tfrac{1}{2}\hbar \begin{pmatrix}\omega_0 & \alpha \\ \alpha & -\omega_0\end{pmatrix} \Rightarrow$$

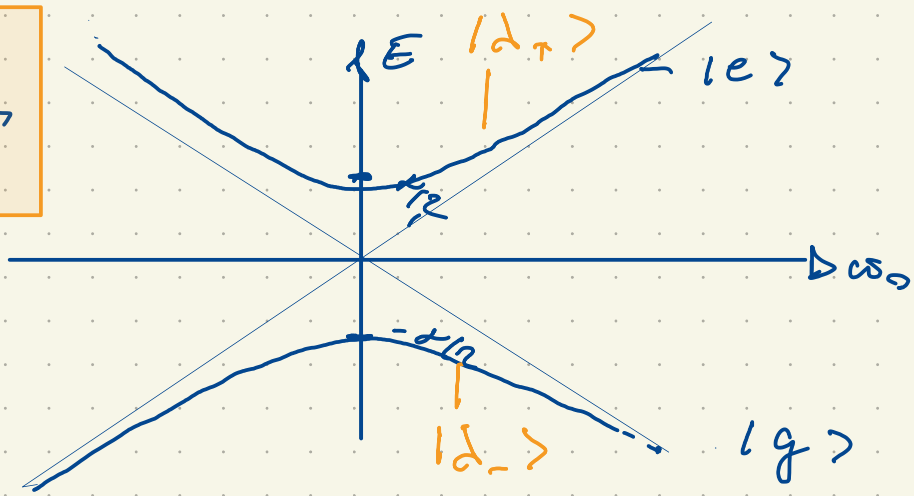
eigenvalues: $\lambda_\pm = \pm\sqrt{\omega_0^2 + \alpha^2}$

eigenstates: @ $\omega_0 = 0$

$$|\lambda_\pm(\omega_0 = 0)\rangle = \frac{1}{\sqrt{2}}\left(|e\rangle \pm |g\rangle\right)$$

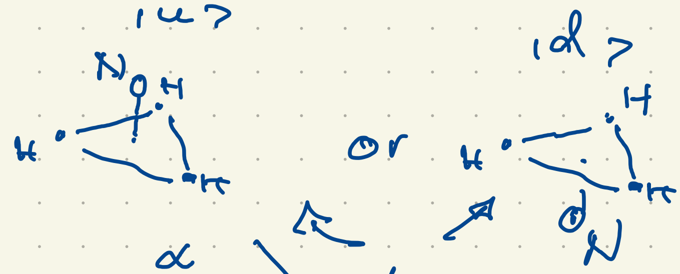
$\boxed{\text{"avoided crossing"}}$

"anti-crossing"

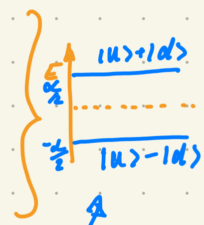


Examples:

Ammonia NH_3



$$H = \frac{1}{2}\hbar \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}$$ degenerate



split by tunneling rate α

Note: same eigenstate superpositions: $\frac{1}{\sqrt{2}}(|u\rangle \pm |d\rangle)$, even if they have same energy

time evolution:

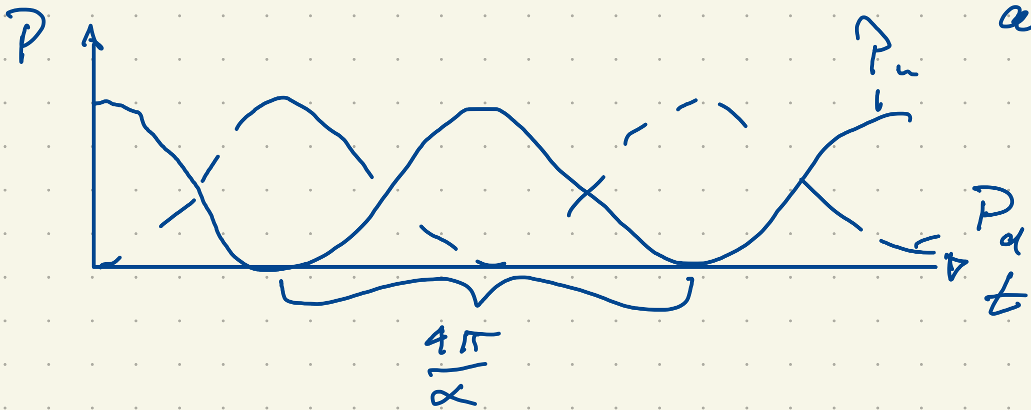
$(|\psi(0)\rangle = |u\rangle)$

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}Ht}\,|\psi(0)\rangle$$

$$= \cos\frac{\alpha}{2}t\,|u\rangle - i\sin\frac{\alpha}{2}t\,|d\rangle$$

$$P_{u/d}(t) = |\langle u/d|\psi(t)\rangle|^2$$

$\left(\frac{\alpha}{2\pi} \approx 24\,\text{GHz}\right.$ important in astrophysics$\left.\right)$



Other examples:

- H_2^+ :

- spin in $\vec{B}$-field

d) Spin 1/2 – generic two-level systems

- Eigenstates

$$S_z |e/g\rangle = \pm \frac{\hbar}{2} |e/g\rangle \Rightarrow S_z = \frac{\hbar}{2}\left(|e\rangle\langle e| - |g\rangle\langle g|\right)$$

$$|e\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |g\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow S_z = \frac{\hbar}{2} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\sigma_z}$$

$$S^2 |e/g\rangle = \hbar^2 \frac{1}{2}\left(\frac{1}{2}+1\right)|e/g\rangle = \frac{3}{4}\hbar^2 |e/g\rangle$$

$$\Rightarrow S^2 = \frac{3}{4}\hbar \, \mathbb{1}$$

Other spin projections

$$S_x = \frac{\hbar}{2}\left(|e\rangle\langle g| + |g\rangle\langle e|\right) \rightarrow \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar}{2}\sigma_x$$

$$S_y = i\frac{\hbar}{2}\left(-|e\rangle\langle g| + |g\rangle\langle e|\right) \rightarrow \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\hbar}{2}\sigma_y$$

$$S_+ = S_x + iS_y = \hbar |e\rangle\langle g| = \frac{\hbar}{2}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

$$S_- = S_x - iS_y = \hbar |g\rangle\langle e| = \frac{\hbar}{2}\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$$

$$S_+|g\rangle = \frac{\hbar}{2}|e\rangle \qquad S_-|e\rangle = \frac{\hbar}{2}|g\rangle$$

$$S_+|e\rangle = 0 \qquad S_-|g\rangle = 0$$

"raising & lowering operators"

$$\left.\begin{aligned} S_x |S_{x\pm}\rangle &= \pm\frac{\hbar}{2}\frac{1}{\sqrt{2}}\left(|e\rangle \pm |g\rangle\right) \\ S_y |S_{y\pm}\rangle &= \pm\frac{\hbar}{2}\frac{1}{\sqrt{2}}\left(|e\rangle \pm i|g\rangle\right) \end{aligned}\right\} \text{eigenstates of } S_x, S_y$$

Also: $[S_i, S_j] = i\hbar \epsilon_{ijk} S_k$ or $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$

$\{S_i, S_j\} = \hbar^2/2 \, \delta_{ij}$ or $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$

Q: bosons, fermions, or other?

A: never bosons - no TLS can be excited twice!

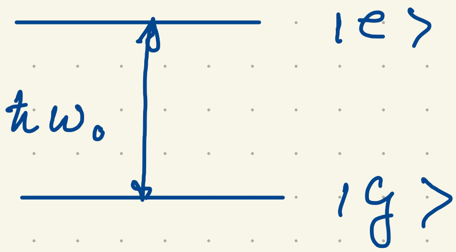
e) Transitions

Why TLS?

- typically, only 2 states of atom are addressed by a laser or similar!

- more complex systems usually use TLSs as basic building blocks.

(i) Interaction picture



$$\begin{pmatrix} |+\rangle, & |1\rangle & \ldots \\ |-\rangle & |0\rangle & \ldots \end{pmatrix}$$

$$\left.\begin{aligned} H_0|e\rangle &= \frac{\hbar}{2}\omega_0 |e\rangle \\ H_0|g\rangle &= -\frac{\hbar}{2}\omega_0 |g\rangle \end{aligned}\right\} H_0 = \frac{\hbar}{2}\omega_0 \sigma_z$$

Transition

$\langle g|V|g\rangle$

$V(t)$: $H = H_0 + V(t)$: $V_{gg} = V_{ee} = 0$

Schrödinger Eq:

$i\hbar \partial_t |\psi\rangle = (H_0 + V)|\psi\rangle$ (1)

know (def): $H_0|n\rangle \equiv \hbar\omega_n |n\rangle$ (general case)

$|\psi\rangle = \sum_n c_n(t)\, e^{-i\omega_n t} |n\rangle$ (2)

(2) mit (1)

$$\langle n| i\hbar \sum_m (\dot{c}_m - i\omega_m c_m) e^{-i\omega_m t} |m\rangle =$$

$$\langle n| \sum_m c_m \hbar \omega_m e^{-i\omega_m t} |m\rangle + \langle n| \sum_m V c_m e^{-i\omega_m t} |m\rangle$$

$$i\hbar \dot{c}_n = \sum_m e^{-i(\omega_m - \omega_n)t} \underbrace{V_{nm}}_{\langle n|V|m\rangle}$$

TLS:

$$\boxed{\begin{aligned} i\hbar \dot{c}_e &= c_g e^{i\omega_0 t} V_{eg} \\ i\hbar \dot{c}_g &= c_e e^{-i\omega_0 t} V_{ge} \end{aligned}}$$