The Bloch Sphere

If we have a TLS (e.g., a qubit), we can write a general state as a superposition of the two states:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \quad , \quad \alpha, \beta \in \mathbb{C}$$

However, the global phase is arbitrary, leaving only 3 parameters.

So, we can visualize any state $|\psi\rangle$ in 3 dimensions.

We should also constrain the normalization of $|\psi\rangle$, such that:

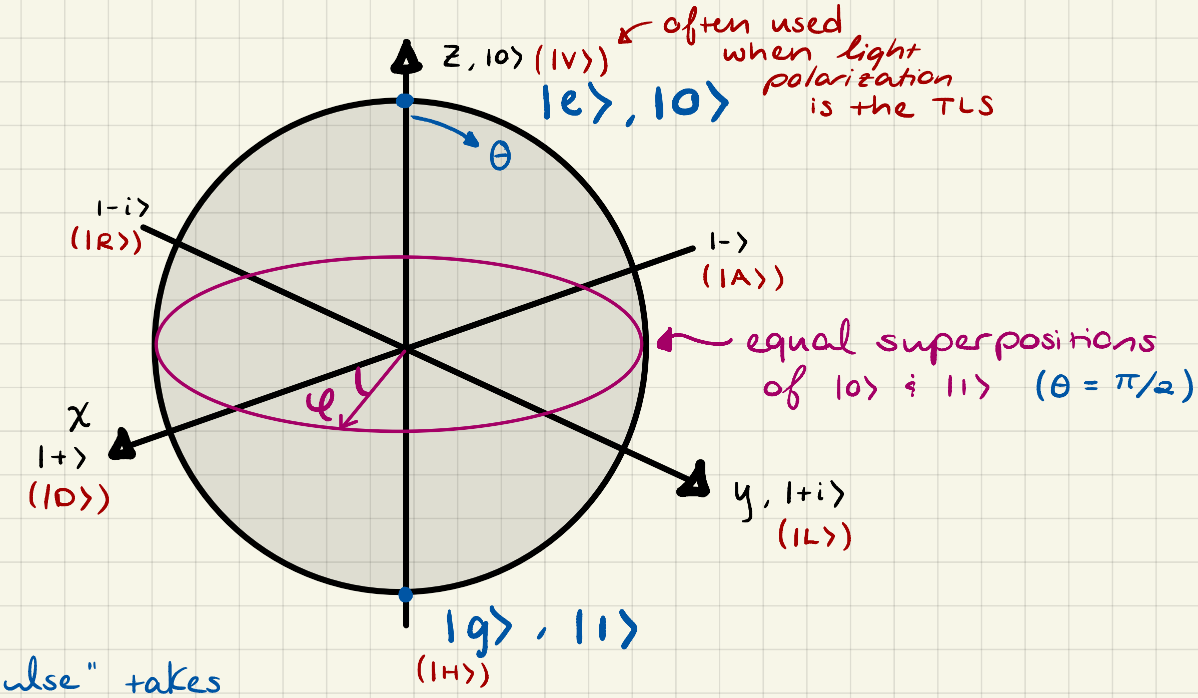
$$\langle\psi|\psi\rangle = |\alpha|^2 + |\beta|^2 = 1$$

Using this, we can write:

$$|\psi\rangle = \left(\cos(\theta/2)|0\rangle + \sin(\theta/2)\underbrace{e^{i\varphi}}_{\text{relative phase}}|1\rangle\right)$$

Since the "length" of $|\psi\rangle$ is always one (for a pure state), it makes sense to encode this as a point on a sphere with radius one.

NOTE: This picture can be extended to the case of mixed states, with purity < 1. In this case, the point lies on the interior of the sphere.



often used when light polarization is the TLS

equal superpositions of $|0\rangle$ & $|1\rangle$ ($\theta = \pi/2$)

A "π-pulse" takes
$\theta \rightarrow \theta + \pi$

We can encode our state $|\psi\rangle$ as a Bloch vector:

$$\langle \vec{\sigma} \rangle = \langle \sigma_x \rangle \hat{x} + \langle \sigma_y \rangle \hat{y} + \langle \sigma_z \rangle \hat{z}$$

$$= \cos\phi \sin\theta \, \hat{x} + \sin\phi \sin\theta \, \hat{y} + \cos\theta \, \hat{z}$$

NOTE:
for mixed states,
this looks like:

$$\rho = \tfrac{1}{2}\left(I + \underset{\uparrow}{\vec{a}} \cdot \underset{\uparrow}{\vec{\sigma}}\right)$$

"Bloch vector" "Pauli vector"

$|\vec{a}| = 1$ for a pure state

We can express the time evolution of our state $|\psi\rangle$ by finding $\langle \dot{\vec{\sigma}} \rangle$.

Our Hamiltonian of interest, in terms of Pauli operators:

$$H = H_0 + V(t)$$

$$H_0 = \frac{\hbar}{2}\omega_0 \sigma_z \quad \text{s.t.} \quad H_0|e\rangle = \frac{\hbar}{2}\omega_0|e\rangle, \quad H_0|g\rangle = -\frac{\hbar}{2}\omega_0|g\rangle$$

$$V(t) = \frac{\hbar}{2}\begin{bmatrix} 0 & \Omega_R e^{-i\omega t} \\ \Omega_R^* e^{i\omega t} & 0 \end{bmatrix}$$

$$= \frac{\hbar}{2}|\Omega_R|\begin{bmatrix} 0 & e^{-i(\omega t-\theta)} \\ e^{i(\omega t-\theta)} & 0 \end{bmatrix}$$

$|\Omega_R| e^{i\theta} := \Omega_R$

$$= \frac{\hbar}{2}|\Omega_R|\begin{bmatrix} 0 & \cos(\omega t-\theta) - i\sin(\omega t-\theta) \\ \cos(\omega t-\theta) + i\sin(\omega t-\theta) & 0 \end{bmatrix}$$

$$= \frac{\hbar}{2}|\Omega_R|\left(\cos(\omega t-\theta)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \sin(\omega t-\theta)\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\right)$$

$$= \frac{\hbar}{2}|\Omega_R|\left(\sigma_x \cos(\omega t-\theta) + \sigma_y \sin(\omega t-\theta)\right)$$

$$\therefore \; H = \frac{\hbar}{2}\omega_0 \sigma_z + \frac{\hbar}{2}|\Omega_R|\left(\sigma_x \cos(\omega t-\theta) + \sigma_y \sin(\omega t-\theta)\right)$$

Using the Ehrenfest theorem:

$$\frac{d\langle \vec{\sigma} \rangle}{dt} = \frac{i}{\hbar}\langle [H, \vec{\sigma}] \rangle + \cancel{\left\langle \frac{\partial \vec{\sigma}}{\partial t} \right\rangle}$$

0 (no explicit time dependence

Pauli algebra: $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$

$$\frac{d\langle\sigma_x\rangle}{dt} = \frac{i}{\hbar}\langle[H,\sigma_x]\rangle$$

$$= \frac{i}{\hbar}\left\langle \frac{\hbar}{2}\omega_0[\sigma_z,\sigma_x] + \frac{\hbar}{2}|\Omega_R|\Big([\sigma_x,\sigma_x]\cos(\omega t-\theta) + [\sigma_y,\sigma_x]\sin(\omega t-\theta)\Big)\right\rangle$$

$$= i\,\langle \omega_0 i\sigma_y - |\Omega_R|\, i\sigma_z \sin(\omega t-\theta)\rangle$$

$$= \frac{1}{2}|\Omega_R|\sin(\omega t-\theta)\langle\sigma_z\rangle - \frac{1}{2}\omega_0\langle\sigma_y\rangle$$

$$\frac{d\langle\sigma_y\rangle}{dt} = \frac{i}{\hbar}\langle[H,\sigma_y]\rangle$$

$$= \frac{i}{\hbar}\left\langle \frac{\hbar}{2}\omega_0[\sigma_z,\sigma_y] + \frac{\hbar}{2}|\Omega_R|\Big([\sigma_x,\sigma_y]\cos(\omega t-\theta) + [\sigma_y,\sigma_y]\sin(\omega t-\theta)\Big)\right\rangle$$

$$= i\,\langle -\omega_0 i\sigma_x + |\Omega_R|\, i\sigma_z \cos(\omega t-\theta)\rangle$$

$$= \omega_0\langle\sigma_x\rangle - |\Omega_R|\cos(\omega t-\theta)\langle\sigma_z\rangle$$

$$\frac{d\langle\sigma_z\rangle}{dt} = \frac{i}{\hbar}\langle[H,\sigma_z]\rangle$$

$$= \frac{i}{\hbar}\left\langle \frac{\hbar}{2}\omega_0[\sigma_z,\sigma_z] + \frac{\hbar}{2}|\Omega_R|\Big([\sigma_x,\sigma_z]\cos(\omega t-\theta) + [\sigma_y,\sigma_z]\sin(\omega t-\theta)\Big)\right\rangle$$

$$= i\,\Big\langle |\Omega_R|\big(-i\sigma_y\cos(\omega t-\theta) + i\sigma_x\sin(\omega t-\theta)\big)\Big\rangle$$

$$= |\Omega_R|\Big(\langle\sigma_y\rangle\cos(\omega t-\theta) - \langle\sigma_x\rangle\sin(\omega t-\theta)\Big)$$

Defining $\vec{\xi} = \big[\,|\Omega_R|\cos(\omega t-\theta),\ |\Omega_R|\sin(\omega t-\theta),\ \omega_0\,\big]^T$, these equations become:

$$\left.\begin{aligned}
\langle\dot{\sigma}_x\rangle &= \xi_y\langle\sigma_z\rangle - \xi_z\langle\sigma_y\rangle \\
\langle\dot{\sigma}_y\rangle &= \xi_z\langle\sigma_x\rangle - \xi_x\langle\sigma_z\rangle \\
\langle\dot{\sigma}_z\rangle &= \xi_x\langle\sigma_y\rangle - \xi_y\langle\sigma_x\rangle
\end{aligned}\right\}\quad \frac{d}{dt}\langle\vec{\sigma}\rangle = \vec{\xi}\times\langle\vec{\sigma}\rangle$$

$$\langle\vec{\sigma}\rangle = \begin{bmatrix}\langle\sigma_x\rangle\\ \langle\sigma_y\rangle\\ \langle\sigma_z\rangle\end{bmatrix} \qquad \xi = \begin{bmatrix}|\Omega_R|\cos(\omega t-\theta)\\ |\Omega_R|\sin(\omega t-\theta)\\ \omega_0\end{bmatrix}$$

Classical mechanics: $\vec{v}_\perp = \vec{\omega}\times\vec{r}$ ← "angular velocity"

velocity perp. to $\vec{r}$

angular velocity

even if there is no driving ($\Omega_R = 0$), the system rotates ("precesses") around $\hat{z}$.

Rabi Oscillations

On resonance $(\Delta = 0)$: $$H = \frac{\hbar}{2}\begin{bmatrix} 0 & \Omega_R \\ \Omega_R^* & 0 \end{bmatrix}$$

Equations of motion:

$$i\hbar \begin{bmatrix} \dot{c}_e \\ \dot{c}_g \end{bmatrix} = H \begin{bmatrix} c_e \\ c_g \end{bmatrix} \rightarrow \begin{aligned} i\dot{c}_e &= \frac{\Omega_R}{2} c_g \\ i\dot{c}_g &= \frac{\Omega_R^*}{2} c_e \end{aligned}$$

$$\left.\begin{aligned} \frac{d}{dt}\left(i\dot{c}_e = \frac{\Omega_R}{2} c_g \right) &\rightarrow i\ddot{c}_e = \frac{\Omega_R}{2} \dot{c}_g \\ i\dot{c}_g = \frac{\Omega_R^*}{2} c_e &\rightarrow \dot{c}_g = -i\frac{\Omega_R^*}{2} c_e \end{aligned}\right\} \quad \begin{aligned} \ddot{c}_e &= -i\frac{\Omega_R}{2}\left(-i\frac{\Omega_R^*}{2} c_e\right) \\ &= -\frac{|\Omega_R|^2}{4} c_e \end{aligned}$$

What solves $\ddot{x}(t) \propto -x(t)$? $\sin(t)$ and $\cos(t)$!

$$c_e(t) = A\cos(|\Omega_R|/2\, t) + B\sin(|\Omega_R|/2\, t)$$

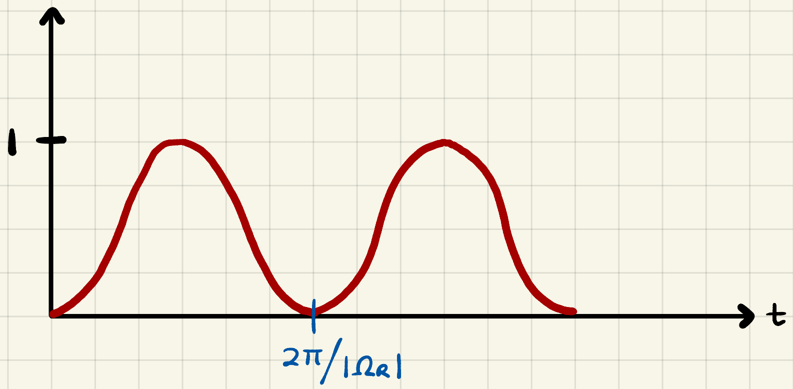
If we assume $c_g(0) = 1$ (system begins in ground state), then:

$c_e(0) = 0 \rightarrow A = 0,\ B = 1$ (from normalization).

$$c_e(t) = \sin(|\Omega_R|/2\, t) \rightarrow c_g(t) = i\frac{2}{\Omega_R}\dot{c}_e = i\frac{|\Omega_R|}{\Omega_R}\cos(|\Omega_R|/2\, t)$$

Now, the population is given by the modulus square:

$$P_e(t) = \sin^2(|\Omega_R|/2\ t) \underset{\substack{\uparrow \\ \text{half-angle} \\ \text{formula}}}{=} \frac{1}{2}\left(1 - \cos(|\Omega_R| t)\right)$$ → *Rabi Oscillations*



Light Shifts and Dressed States

Big idea: By driving the TLS, we change the Hamiltonian. This causes the energies to shift (known as light shifts in the limit of large detuning) and the eigenstates to change (dressed states).

Hamiltonian with a laser field (in the rotating frame + RWA):

$$H = \hbar \begin{bmatrix} -\Delta & \Omega_R/2 \\ \Omega_R^*/2 & 0 \end{bmatrix}$$

NOTE: script uses other convention for $\Omega_R \to -\Omega_R$

Diagonalize: $$(-\Delta-\lambda)(-\lambda) - |\Omega_R|^2/4 = 0$$

$$\lambda^2 + \Delta\lambda - |\Omega_R|^2/4 = 0$$

$$\lambda_\pm = -\frac{1}{2}\left(\Delta \pm \sqrt{\Delta^2 + |\Omega_R|^2}\right)$$

$$\begin{bmatrix} -\Delta-\lambda_+ & \Omega_R/2 \\ \Omega_R^*/2 & -\lambda_+ \end{bmatrix} |\lambda_+\rangle = \vec{0} \rightarrow \begin{bmatrix} -\Delta/2 + \sqrt{\cdot}/2 & \Omega_R/2 \\ \Omega_R^*/2 & \Delta/2 + \sqrt{\cdot}/2 \end{bmatrix} |\lambda_+\rangle = \vec{0}$$

$$|\lambda_+\rangle = \mathcal{N}_+ \begin{bmatrix} -\Omega_R \\ -\Delta + \sqrt{\cdot} \end{bmatrix}, \quad \mathcal{N}_+ = \left[|\Omega_R|^2 + (-\Delta + \sqrt{\cdot})^2\right]^{-1/2}$$

check: $$-(-\Delta+\sqrt{\cdot})\Omega_R + (\Omega_R)(-\Delta+\sqrt{\cdot}) = 0 \quad \checkmark$$

$$-(\Omega_R^*)\Omega_R + (\Delta+\sqrt{\cdot})(-\Delta+\sqrt{\cdot}) = -|\Omega_R|^2 - \Delta^2 + (\sqrt{\cdot})^2 = 0 \quad \checkmark$$

$$\begin{bmatrix} -\Delta-\lambda_- & \Omega_R/2 \\ \Omega_R^*/2 & -\lambda_- \end{bmatrix} |\lambda_-\rangle = \vec{0} \rightarrow \begin{bmatrix} -\Delta/2 - \sqrt{\cdot}/2 & \Omega_R/2 \\ \Omega_R^*/2 & \Delta/2 - \sqrt{\cdot}/2 \end{bmatrix} |\lambda_-\rangle = \vec{0}$$

$$|\lambda_-\rangle = \mathcal{N}_- \begin{bmatrix} -\Delta + \sqrt{\cdot} \\ \Omega_R^* \end{bmatrix}, \quad \mathcal{N}_- = \left[|\Omega_R|^2 + (-\Delta + \sqrt{\cdot})^2\right]^{-1/2}$$

check: $$(-\Delta-\sqrt{\cdot})(-\Delta+\sqrt{\cdot}) + (\Omega_R)\Omega_R^* = \Delta^2 - (\sqrt{\cdot})^2 + |\Omega_R|^2 = 0 \quad \checkmark$$

✓ $$(\Omega_R^*)(-\Delta+\sqrt{\cdot}) + (\Delta-\sqrt{\cdot})\Omega_R^* = 0 \quad \checkmark$$

$|\lambda_\pm\rangle$ are the dressed states.

We can look at various limits.

On resonance ($\Delta = 0$):

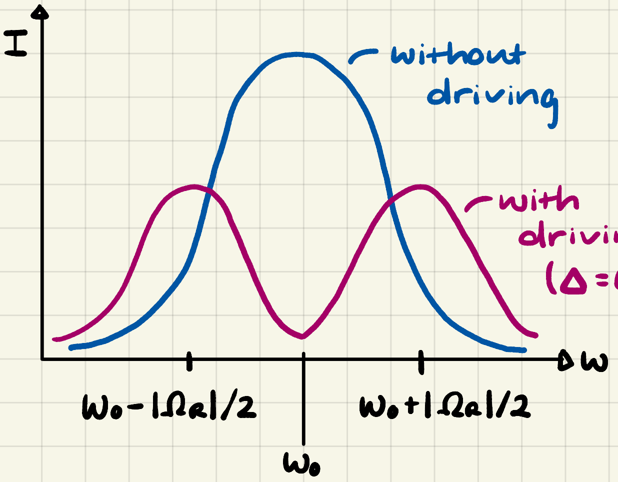
$$|\lambda_+\rangle = \frac{1}{|\Omega_R|\sqrt{2}}\begin{bmatrix} -\Omega_R \\ |\Omega_R| \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -e^{i\phi} \\ 1 \end{bmatrix}$$

$\Omega_R = |\Omega_R| e^{i\phi}$

dressed states are equal superpositions of $|e\rangle$ & $|g\rangle$.

$$|\lambda\rangle = \frac{1}{\sqrt{2}|\Omega_R|}\begin{bmatrix} |\Omega_R| \\ \Omega_R^* \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ e^{-i\phi} \end{bmatrix}$$

$$\lambda_\pm = \mp \frac{|\Omega_R|}{2}$$



For off resonance ($|\Delta| \gg |\Omega_R|$):

$$\sqrt{\Delta^2 + |\Omega_R|^2} \approx |\Delta| + \frac{1}{2}\frac{|\Omega_R|^2}{|\Delta|} \; ; \quad \lambda_\pm \approx -\frac{1}{2}\left(\Delta \pm |\Delta| \pm \frac{1}{2}\frac{|\Omega_R|^2}{|\Delta|}\right)$$

$\Delta > 0$ ("blue detuned"):

$$\lambda_+ \approx -\Delta - \frac{|\Omega_R|^2}{4\Delta}$$

$$\lambda_- \approx \frac{|\Omega_R|^2}{4\Delta}$$

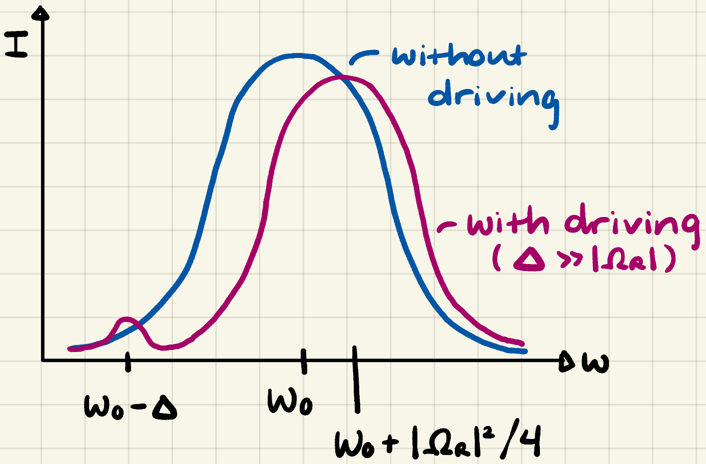
$$|\lambda_+\rangle \approx |e\rangle, \quad |\lambda_-\rangle = |g\rangle$$

$\Delta < 0$ ("red detuned"):

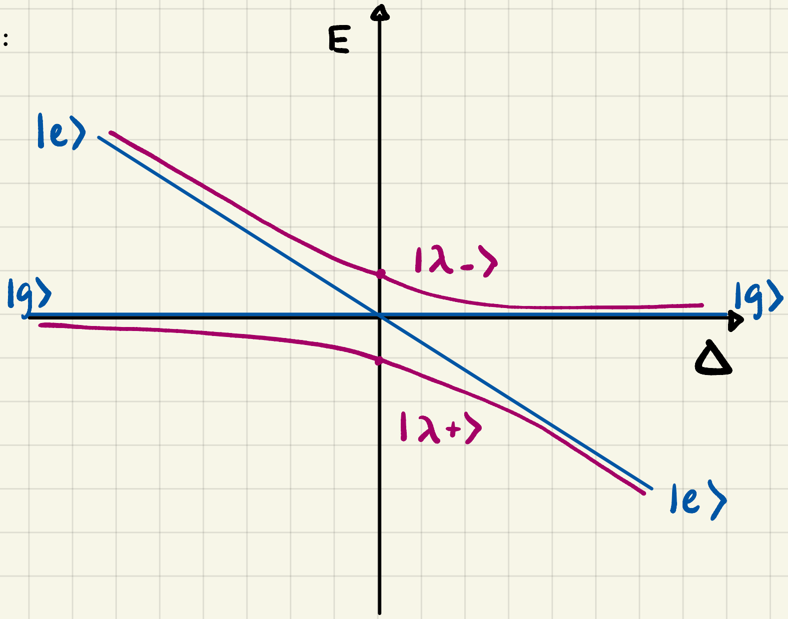
$$\lambda_+ \approx -\frac{|\Omega_R|^2}{4|\Delta|} = \frac{|\Omega_R|^2}{4\Delta}$$

$$\lambda_- = -\Delta - \frac{|\Omega_R|^2}{4\Delta}$$

$$|\lambda_+\rangle \approx |g\rangle, \quad |\lambda_-\rangle \approx |e\rangle$$



Piecing everything together:



The Bloch-Siegert Shift

When the detuning Δ is very large, RWA is not accurate.

We can (approximately) include the counter-rotating terms as another field.

This appears along the digonal of the Hamiltonian (like $\pm\Delta/2$, but now with frequency $\omega+\omega_0$ rather than their difference).

$$H_{BS} = \frac{\hbar}{2}\begin{bmatrix} \omega+\omega_0 & \Omega_R \\ \Omega_R^* & -(\omega+\omega_0) \end{bmatrix}$$

Diagonalize to find the energy shifts:

$$(\omega+\omega_0-\lambda)(-\omega-\omega_0-\lambda) - |\Omega_R|^2 = 0$$

$$\lambda^2 - (\omega+\omega_0)^2 - |\Omega_R|^2 = 0$$

$$\lambda = \pm\sqrt{(\omega+\omega_0)^2 + |\Omega_R|^2}$$

$$= \pm(\omega+\omega_0)\sqrt{1+\varepsilon} \qquad \varepsilon := \frac{|\Omega_R|^2}{(\omega+\omega_0)^2}$$

$$\approx \pm(\omega+\omega_0)\left(1+\frac{\varepsilon}{2}\right)$$

$$= \pm(\omega+\omega_0) \pm \underbrace{\frac{|\Omega_R|^2}{2(\omega+\omega_0)}}$$

$$\Delta E_{BS} = \frac{1}{2}\,\frac{|\Omega_R|^2}{2(\omega+\omega_0)}$$