Lecture 8 — 10/2

Homework: not more than 10-15 hrs max (10 ave)

How to shorten: collaborate

ChatGPT

Ask for help

⋮

system + environment:

$$\dot{\rho}_{sys} = \frac{1}{i\hbar}\left[H_{sys} + \cancel{H_{envir.}} + H_{sys-env}, \rho_{sys}\right]$$

0

commutes w/ ρ_{sys}

<u>Interaction picture</u>

$$U_0 = e^{-\frac{i}{\hbar}H_0 t} \; : \; \tilde{\rho} = U_0^+ \rho U_0 \; , \; \tilde{H}_{int} = U_0^+ H_{int} U_0$$

for any $H = H_0 + H_{int}$

here : $H_0 \hat{=} H_{sys} + H_{env}$; $H_{int} \hat{=} H_{sys-env}$.

$$\Rightarrow U_0 = \exp\left[-\frac{i}{\hbar}\left(H_{sys} + H_{env}\right)\right]$$

$$\dot{\tilde{\rho}}_{sys} = \frac{1}{i\hbar}\left[\tilde{H}_{sys-env}, \tilde{\rho}_{sys}\right]$$

c) Kraus operators & Lindblad Eq.

$\rho_{s(ys)} = Tr_{E(nv)}\, \rho_{total}$; assume: $\rho_{total} = \rho_s \otimes |e\rangle\langle e|$ at some time t_0

Universe (ρ_{total}) evolves with

U_{SE} ; basis of E: $|e_k\rangle$ $\qquad (|e\rangle = \sum_k c_{ek} |e_k\rangle)$

$$\rho_s' = Tr_E\, U_{SE}\left(\rho_s \otimes |e\rangle\langle e|\right) U_{SE}^+$$

$$\sum_k \underbrace{\langle e_k | U_{SE} | e\rangle}\, \rho_s\, \underbrace{\langle e | U_{SE}^+ | e_k}$$

$$\equiv \sum_k M_k \rho_s M_k^+$$

M_k has dimension of system!

<u>Properties:</u>

- M_k is system operator
- $\sum_k M_k^+ M_k = \sum_k \langle e | U_{SE}^+ | e_k\rangle\langle e_k | U_{SE} | e\rangle =$
  $= \langle e | U_{SE}^+ U_{SE} | e\rangle = \mathbb{1}_S$

M_k: "Kraus operators"

$$\rho_S' = \sum_k M_k \rho_S M_k^+ \qquad \text{"Kraus representation" of linear map}$$

- Time evolution: (drop "S")

"$\rho' \overset{?}{\to} \dot{\rho}$": use time evolution operator (U_{SE})

$\rho(t) \longrightarrow \rho(t+\delta t)$ is special case of linear map

$$\underset{\rho(t)+\dot{\rho}(t)\cdot\delta t}{\rho(t+\delta t)} = \underbrace{\sum_k M_k \rho(t) M_k^+}_{\Rightarrow \text{ linear in } \delta t\,!} \Rightarrow \mathbb{1}\cdot\rho(t) + \mathcal{O}(\delta t)$$

$\Rightarrow M_k = \mathbb{1} + \mathcal{O}(\delta t)$ or

$M_k = \mathcal{O}(\sqrt{\delta t})$

$\Rightarrow$ define (most general case):

$$M_0 = \mathbb{1} + (K + iH)\delta t \qquad (K, H \text{ Hermitian}$$

$$M_k = L_k \sqrt{\delta t} \quad (\text{for } k \neq 0)$$

then $\sum_k M_k \rho M_k^+ =$

$$(\mathbb{1} + (K+iH)\delta t)\rho(\mathbb{1} + (K-iH)\delta t) + \sum_{k\neq 0} L_k \rho L_k^+ \delta t =$$

$$= \rho(t) + \underbrace{\left[(K\rho + \rho K) + i[H,\rho] + \sum_{k\neq 0} L_k \rho(t) L_k^+\right]}_{\dot{\rho}(t)}\delta t$$

remember:

$$\sum_k M_k^+ M_k = \mathbb{1} + \underbrace{\left[(K+\cancel{iH}) + (K-\cancel{iH}) + \sum_{k\neq 0} L_k^+ L_k\right]}_{\overset{!}{=} 0}\delta t$$

$$\Rightarrow K = -\frac{1}{2}\sum_{k\neq 0} L_k^+ L_k$$

$$\boxed{\dot{\rho} = i[H,\rho] + \frac{1}{2}\sum_{k\neq 0}\left(2L_k\rho L_k^+ - L_k^+ L_k\rho - \rho L_k^+ L_k\right)}$$

"Master Equation"

most general EOM for ρ?

where dimension:

$$[H] \equiv \frac{1}{s}$$

$$[L] = \frac{1}{\sqrt{s}}$$

Thus: $H \rightarrow -\frac{1}{\hbar} H$ (for H to be Hamiltonian)

"<u>Lindblad Operator</u>":

$$\mathcal{L}[\rho] \equiv \frac{1}{2} \sum_{k \neq 0} \left(2 L_k \rho L_k^{+} - L_k^{+} L_k \rho - \rho L_k^{+} L_k \right)$$

"<u>Lindblad Equation</u>":

$$\boxed{\dot{\rho} = \mathcal{L}[\rho]}$$

<u>Master Eq:</u>

$$\boxed{\dot{\rho} = \frac{1}{i\hbar} [H, \rho] + \mathcal{L}[\rho]}$$

Master Eq:

$$\dot{\rho} = \frac{1}{i\hbar}[H, \rho] + \mathcal{L}[\rho]$$

with the Lindblad term

$$\mathcal{L}[\rho] = \frac{1}{2}\sum_k \underbrace{\left(2L_k \rho L_k^+ - L_k^+ L_k \rho - \rho L_k^+ L_k\right)}_{\equiv \tilde{\mathcal{L}}_k}$$

For TLS:

Possible system operators for the L_k are the Pauli matrices $(\sigma_-, \sigma_+, \sigma_z)$

$\underline{\sigma_-}$: $L_k = \sqrt{\gamma}\,|g\rangle\langle e|$ $\Rightarrow$

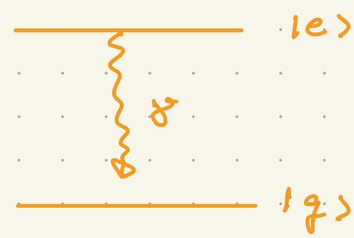
$$\tilde{\mathcal{L}}_k = 2|g\rangle\langle e|\rho|e\rangle\langle g| - |e\rangle\langle e|\rho - \rho|e\rangle\langle e|$$

$$\Rightarrow \dot{\rho}_{ee}\big|_{\mathcal{L}_k} = \gamma\,\langle e|\tilde{\mathcal{L}}_k|e\rangle = -\gamma\rho_{ee}$$

similar: $\dot{\rho}_{gg}\big|_{\mathcal{L}_k} = \gamma\rho_{ee}$

$$\dot{\rho}_{eg}\big|_{\mathcal{L}_k} = -\frac{\gamma}{2}\rho_{eg}$$

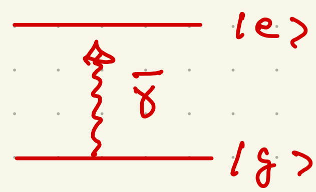
$$\dot{\rho}_{ge}\big|_{\mathcal{L}_k} = -\frac{\gamma}{2}\rho_{ge}$$



decay ($|e\rangle \to |g\rangle$)

$\underline{\sigma_+}$: $L_k = \sqrt{\bar{\gamma}}\,|e\rangle\langle g|$ $\Rightarrow$

$$\dot{\rho}_{ee}\big|_{\mathcal{L}_k} = \bar{\gamma}\rho_{gg}; \quad \dot{\rho}_{gg}\big|_{\mathcal{L}_k} = -\bar{\gamma}\rho_{gg}; \quad \dot{\rho}_{eg}\big|_{\mathcal{L}_k} = -\frac{\bar{\gamma}}{2}\rho_{eg}; \quad \dot{\rho}_{ge}\big|_{\mathcal{L}_k} = -\frac{\bar{\gamma}}{2}\rho_{ge}$$



"pump"

Typically: $\gamma \to (\bar{n}+1)\Gamma$: $\bar{n}$ = number of thermal photons (≈ 0 for typical atomic transitions)

Γ : spontaneous emission rate

$$\bar{\gamma} \to \bar{n}\Gamma$$

σ_z: $L_k = \sqrt{\gamma_{deph}}\,(|e\rangle\langle e| - |g\rangle\langle g|)$

$$\Rightarrow \left.\begin{aligned} \dot\rho_{ee}\big|_{\mathcal{L}_k} &= 0 \\ \dot\rho_{gg}\big|_{\mathcal{L}_k} &= 0 \\ \dot\rho_{eg}\big|_{\mathcal{L}_k} &= -\gamma_{deph}\,\rho_{eg} \\ \dot\rho_{ge}\big|_{\mathcal{L}_k} &= -\gamma_{deph}\,\rho_{ge} \end{aligned}\right\}$$

This type of decay is known as "dephasing": It does not affect the energy of the system, just the phase/coherence!

Literature for derivation of master equation w/ Kraus operators:

e.g. John Preskill lecture notes "Quantum Information" chapter 3 (on Preskill Caltech website)

$\Rightarrow$ full master equation for TLS:

$$\frac{H}{\hbar} = -\delta|e\rangle\langle e| - \frac{\Omega}{2}|e\rangle\langle g| - \frac{\Omega^*}{2}|g\rangle\langle e|$$

$$\Rightarrow -i\left[\frac{H}{\hbar},\rho\right] = i\delta\left[|e\rangle\langle e|,\rho\right] + i\frac{\Omega}{2}\left[|e\rangle\langle g|,\rho\right] + i\frac{\Omega^*}{2}\left[|g\rangle\langle e|,\rho\right]$$

$\Rightarrow \dot\rho_{ee} = \langle e|\dot\rho|e\rangle$

$$\boxed{\begin{aligned} \dot\rho_{ee} &= -\gamma\rho_{ee} + \frac{i}{2}(\Omega\rho_{ge} - \Omega^*\rho_{eg}) \\ \cancel{\dot\rho_{gg}} &\cancel{= -\dot\rho_{ee}} \qquad \rho_{ee} + \rho_{gg} = 1 \\ \dot\rho_{eg} &= -\left(\frac{\gamma}{2} + \gamma_{deph} - i\delta\right)\rho_{eg} - \frac{i}{2}\Omega(\rho_{ee} - \rho_{gg}) \\ \dot\rho_{ge} &= (\dot\rho_{eg})^* \end{aligned}}$$

d) Saturation (Example)

solve in steady state: $\dot{\rho}_{ij} = 0$

for TLS: ($\gamma_{deph} = 0$):

$$\Rightarrow \quad \boxed{\rho_{ee} = \frac{|\Omega_R|^2}{4} \frac{1}{\delta^2 + \left(\frac{\gamma}{2}\right)^2 + \frac{1}{2}|\Omega_R|^2}}$$

rate of scattered light (fluorescence):

$$\gamma_{scatt} = \gamma \rho_{ee}$$

(i) low intensity: $|\Omega_R| \ll \gamma$

$$\rho_{ee} \simeq \frac{|\Omega_R|^2}{4} \frac{1}{\delta^2 + (\gamma/2)^2} \qquad \leftarrow \text{cf. earlier result}$$

$$\leq \left|\frac{\Omega_R}{\gamma}\right|^2$$

(ii) high intensity: $|\Omega_R| \gg \gamma$

$$\rho_{ee} \approx \frac{1}{2} \frac{1}{1 + 2\,\delta^2/|\Omega_R|^2} \xrightarrow{\delta \to 0} \frac{1}{2}$$

"saturated": $s = \dfrac{|\Omega_R|^2}{\gamma^2}$

"saturation parameter"