Physics 285a Problem Set 4

posted October 3, 2023, due October 11, 2023

Problem 1. Show that any density operator defined as

$$\rho \;=\; \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right|$$

for any set of normalized (but not necessarily orthogonal) states $\{|\psi_i\rangle\}$ can also be written as

$$\rho \;=\; \sum_n p'_n \left|n\right\rangle \left\langle n\right|,$$

where $\{|n\rangle\}$ is an orthonormal basis, and the p'_n are also probabilities. For that,

a) Start by showing that this is true (and what the p'_n and $\{|n\rangle\}$ are) for a density matrix $\rho = \frac{1}{2}\left|\psi_1\right\rangle\left\langle\psi_1\right| + \frac{1}{2}\left|\psi_2\right\rangle\left\langle\psi_2\right|$ with $|\psi_1\rangle = |1\rangle$ and $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. One very straightforward way to do this (but one that does not directly help with part (b)) is to just diagonalize ρ and show that the eigenvalues are all positive.

b) Then show in general.

If you have an idea about the general proof, feel free to do these in reverse order.

Problem 2. Given an (entangled) state $|\phi\rangle \;\equiv\; (|01\rangle - |10\rangle)/\sqrt{2}$ of two two-level systems, what is the state of either one of its individual subsystems?

Problem 3. Take a two-level system with the states $|\pm\rangle$. Write down the density operators in matrix form corresponding to the superposition state $|\psi\rangle \;=\; \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right)$ and to the 50/50 mixture of the states $|+\rangle$ and $|-\rangle$.

Problem 4. Show that:

$$p_m \;=\; \mathrm{Tr}\Big(P_m \rho P_m\Big),$$

where p_m is the probability of being in eigenstate $|m\rangle$ and P_m is the projector onto this state.

Problem 5. Assume a three level system (call them $|1\rangle, |2\rangle, |3\rangle$): All three levels have different energies. The upper two are coupled by a driving field, and there is decay from the upper to the middle and from the middle to the lower (with different decay rates. Write down the equations of motion for this system, i.e., $\dot{\rho}_{11} = \ldots, \dot{\rho}_{12} = \ldots$ and so on.

Problem 6. Electromagnetically-induced transparency (extra credit): We will demonstrate that the mathematical toolboxes developed in class, namely the master equations, are sufficiently powerful for exploring striking quantum optical phenonema such as *electromagnetically-induced transparency* (EIT).

We will study transmission of a *probe* laser beam through a gaseous atomic medium. The atomic response is encoded in the following relations for polarizability

$$P = \epsilon_0 \chi \mathcal{E} = \frac{N}{V} \mathcal{P}_{31} \rho_{31} \tag{1}$$

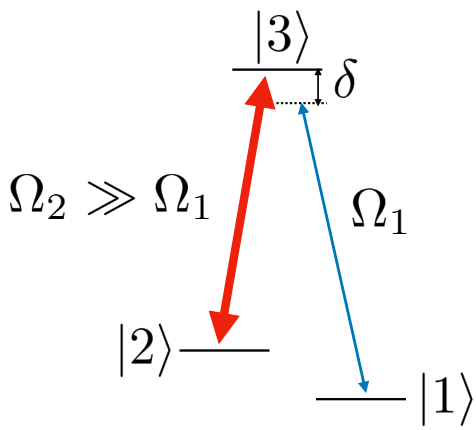

Figure 1: The three-level diagram for EIT. The Rabi frequencies of the probe and pump are Ω_1 and Ω_2 respectively. In practice, we may scan the frequency of the probe to look at e.g. the absorption spectrum while leaving the pump on-resonance.

for a two-level system with states $|1\rangle$ and $|3\rangle$ described by the density matrix ρ_{31}. We are interested in the complex susceptibility $\chi = \chi' + i\chi''$. Its real part describes the dispersive behavior i.e. $n = 1 + \chi'/2$. Its imaginary part is the attenuation factor (or in some cases, amplification).

Consider a three-level system in Fig.(1) described by

$$H = -\delta\,|3\rangle\langle 3| - \delta\,|2\rangle\langle 2| - \Bigg[\Omega_1\,|3\rangle\langle 1| + \Omega_2\,|3\rangle\langle 2| + h.c.\Bigg]. \tag{2}$$

We may also assume that the only decay channels go from $|3\rangle$ to the two ground states. The decay rate is γ for both channels.

(a) Assume only for this part that $\mathcal{P}_{23} = 0, \Omega_2 = 0$ so we are left with a two-level system addressed by the probe field. Calculate χ using the steady-state solution to the master equation for ρ_{31}. Separately plot χ' and χ'' vs. δ/γ for $\Omega_1 = 0.01\gamma$.

(b) Now reintroduce the ground state $|2\rangle$ and the pump field. Write down the master equations for ρ_{31} and ρ_{21} and solve for the steady-state solution for ρ_{31}. What assumptions do you have to make on the various populations when the probe is weak?

(c) Plot χ' and χ'' for $\Omega_2 = 0.5\gamma$ and overlay them on the plots in part (a) for comparison. You should be able to see a sharp drop in absorption in χ''. What is the *transparency window* i.e. the width of the narrow feature?

(d) One particularly striking feature that makes EIT different from tuning the probe out of resonance to recover transparency is the properties of light in the atomic medium. One of these properties is *slow light*, which appears as suppression of the group velocity $\frac{d\omega}{dk}$. Show that the slope of the index of refraction $\frac{dn}{d\omega}$ diverges and argue that the group velocity tends to zero as $\Omega_2 \to 0$.