Physics 285a Problem Set 7

posted October 25, 2023, due November 1, 2023

Problem 1.

Dirac Hamiltonian $H = c\alpha \cdot \mathbf{p} + V(\mathbf{r}) + \beta mc^2$ and angular momentum: In this problem, you will have an opportunity to confirm the Dirac results to the fine structure corrections briefly discussed in class.

(a) An expanded Dirac equation shows that there are three "fine structure" terms that must be added to the Schrodinger Hamiltonian:

$$\begin{aligned} H_{RKE} &= -\frac{\mathbf{p}^4}{8m^3c^2}, &(1)\\ H_{SO} &= \frac{1}{2m^2c^2}\frac{1}{r}\frac{\partial V}{\partial r}\mathbf{L}\cdot\mathbf{S}, &(2)\\ H_D &= \frac{\hbar^2 Ze^2}{8m^2c^2\epsilon_0}\delta(\mathbf{r}). &(3)\end{aligned}$$

Evaluate the matrix element of H_{RKE}, H_{SO} and H_D. Which basis (coupled or uncoupled) is best for which calculation?

Add the three fine structure matrix elements for each of three cases and show that the result only depends on the total angular momentum (and of course, the princical quantum number).

Hint: You may find some of the following identities useful. Let $\rho = r/a_0$, the expection values of various powers of ρ evaluated with $|nlm\rangle$ are

$$\langle\left(\frac{1}{\rho}\right)^3\rangle = \frac{1}{l(l+1/2)(l+1)n^3} \tag{4}$$

$$\langle\left(\frac{1}{\rho}\right)^2\rangle = \frac{1}{(l+1/2)n^3} \tag{5}$$

$$\langle\left(\frac{1}{\rho}\right)\rangle = \frac{1}{n^2} \tag{6}$$

$$\langle\rho\rangle = \frac{3n^2 - l(l+1)}{2} \tag{7}$$

(b) Expand the exact result to the Dirac equation

$$E_{nj}^{Dirac} = mc^2\left[\frac{1}{\sqrt{1+\left(\frac{Z\alpha}{n-j-\frac{1}{2}+\sqrt{(j+\frac{1}{2})^2 - Z^2\alpha^2}}\right)^2}} - 1\right] \tag{8}$$

for the hydrogen atom in powers of $Z\alpha$ to get the leading fine structure terms we derived in class. What happens for a hydrogenic atom with $Z = 140$?