Physics 285a Problem Set 9

posted November 14, 2023, due November 15, 2023

Problem 1. Wigner-Eckart theorem and optical transitions and angular momentum

An atom of total angular momentum F has a spontaneous radiation rate γ that consist of a sum of rates ("Einstein A-coefficients") of all transitions out of F. For example, assume it radiates to a lower level with angular momentum $F' = F - 1$. The problem is to find the rates for the various allowed transitions, i.e., the fraction of the radiation that goes into each of the possible transitions $(F, m) \to (F', m')$. Each of the rates is proportional to $|\langle F, m_F \, | Y_{1,q} | \, F', m'_F \rangle|^2$. The rates can be found by either direct evaluation of matrix elements of by applying the following considerations:

(1) The sum of the rates out of each state F, m must equal A.

(2) The sum of the rates into each state F', m' must equal $A\frac{2F+1}{2F'+1}$.

(3) An upolarized mixture of radiators in level F must emit equal intensities of light with each of the three polarization components $(z, \pm)$. (Why?)

(4) The rate for a transition $(F, m \to F', m')$ must be the same as for $(F, -m \to F', -m')$.

Consider the situation $F = 1, F' = 2$. Designate the transitions by letters as follows:

$$\begin{array}{lrcl} a\text{:} & m = 1 & \to & m' = 2 \\ b\text{:} & m = 1 & \to & m' = 1 \\ c\text{:} & m = 1 & \to & m' = 0 \\ d\text{:} & m = 0 & \to & m' = -1 \\ e\text{:} & m = 0 & \to & m' = -2 \end{array}$$

...

Note that this list of transitions is not exhaustive, and additional transitions may be important.

a) Write down the Wigner-Eckart theorem in your own words, find an outline of the proof anywhere and put this into your own words.

b) Find a definition of the so-called $3j$ symbols and their connections to the Clebsch-Gordan coefficients. What would be one reason to use $3j$ symbols rather than Clebsch-Gordan coefficients?

c) The Wigner-Eckart theorem can be used to evaluate matrix elements in terms of an m-independent quantity. Note, however, that F involves orbital, electron spin, and nuclear spin components. Does the W.-E. theorem as stated still apply? Explain what conditions must be satisfied?

d) Find the rates for a through e in terms of the relevant Einstein-a coefficient (just call it A) using the appropriate version of the Wigner-Eckart theorem and make a figure of your results. (Clebsch-Gordan coefficients can either be worked out from first principles, taken from a table in a quantum mechanics of spectroscopy text, of computed with Mathematica.)

e) Using the symmetry considerations and conservation of probabilities (i. e. the total number of decaying atoms from level F should be equal to the total number arriving in F') show rules (2) and (4) must be true. Find the rates for a through e using rules (1) - (4), and make a figure of your results. *(Note: Obviously, parts (d) and (e) should give the same results!)*

Problem 2. In class, we calculated explicitly the spontaneous emission rate as the imaginary part of the change in population of the upper state in a two-level atoms. Do the same for the real part, i.e., the Lamb shift. You will get a singularity at some point – this is where we earlier in class averaged over the s-wavefunction. You can stop before this averaging step. Do, however, check in the literature/online for ways how to do the calculation from here and briefly describe.