Introduction

Goal

study topological objects up to some equivalence relation

homeomorphism too strong

$A \underset{\text{homeo}}{\not\cong} O$

but A can deform into O

want to ~~classify~~ study things up to deformation $\rightsquigarrow$ homotopy

tool

$\boxed{\text{Topological objs \& maps} \,/\, \text{homotopy } \sim} \rightsquigarrow \boxed{\text{Algebraic invariants}}$

want

① computable ② reflect ~~properties~~ structures

③ invariant wrt homotopy

Ex

① # of components

② genus.

(connected orientable surfaces)

Some definitions

Convention X Y Z : topological spaces.

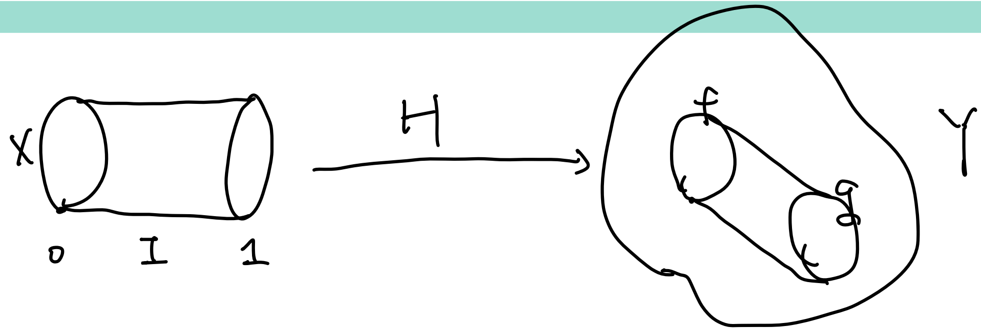
all maps are continuous maps.

Def (homotopic). (write $f \simeq g$)

$f, g: X \longrightarrow Y$ are **homotopic** if

$\exists\, H: X \times I \longrightarrow Y$ s.t. $H|_{X\times\{0\}} = f$.

$I = [0,1]$ $\qquad H|_{X\times\{1\}} = g$

- 

- think of H as a family of maps from X to Y parametrized over I.

  for each $t \in [0,1] \rightsquigarrow H_t: X \to Y$

---

prop $\simeq$ defines an equivalence relation on the set of maps from X to Y.

Def ① $f: X \to Y$ is a **homotopy equivalence** if $\exists\, g: Y \to X$ s.t. $f \circ g \simeq id_Y$.

$g \circ f \simeq id_X$

② when $\exists\, f: X \to Y$ homotopy equivalence, we say X **homotopy equivalent** to Y.

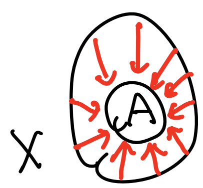
Compare definition with geometric intuition

Fact $X \xrightarrow{f} Y$ homotopy equivalence

$\iff$ X & Y are homeomorphic to a strong deformation retract of another space.

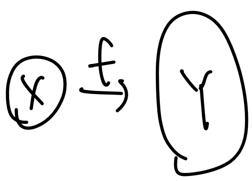
Def (strong deformation retract) $F: X \times I \to X$ is a strong def. retract from X to $\underbrace{\mathrm{Im}(F_1)}_{A}$ if.

$F_0 = \mathrm{id}_X$ & $F|_{A \times \{t\}} = \mathrm{id}_A$.

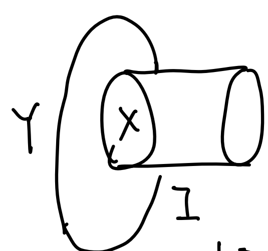


points in A: in same position

points outside A: deformed to A



$X \xrightarrow{f} Y$

$Z =$ 

mapping cylinder

$X \times I \sqcup Y \,/\, \begin{matrix}(x,0) \sim y \\ \text{if } f(x) = y\end{matrix}$

$Z \xrightarrow[\text{deform retract}]{\text{strong}} Y$: smash $X \times I$ to $X \times \{0\}$

$Z \xrightarrow[\text{deform retract}]{\text{strong}} X$: use homotopy equivalence (exercise)

Based world

Based spaces : X with a pre-assigned basept $x_0 : * \to X$, write (X, x_0)

maps of based spaces : preserves base pt

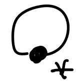
$(X, x_0) \xrightarrow{f} (Y, y_0)$ :: $f : X \to Y$ with $f(x_0) = y_0$.

Def (fundamental grps).

The fundamental group of a based space (X, x_0), denoted by $\pi_1(X, x_0)$, is

① underlying set : homotopy equivalence classes of based maps $(S^1, *) \to (X, x_0)$.
circle

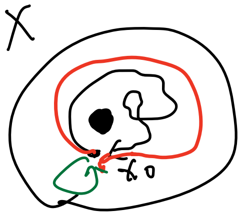
think of a map $(S^1,*) \to (X,x_0)$ as a loop.

$S^1 = I/\{0\}\sim\{1\}$ 

- $(S^1,*) = (I/\sim,\{0\}) \xrightarrow{p} (X,x_0)$

$\rightsquigarrow$ • $p: I \to X$ with

$p(0) = p(1) = x_0$



2 group structure

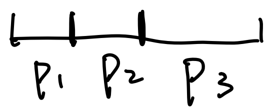
a) addition b) identity c) inverse

a) 

$[p_1] + [p_2] =$  $= t \mapsto \begin{cases} p_1(2t) & t\in[0,\frac{1}{2}] \\ p_2(2t-1) & t\in[\frac{1}{2},1] \end{cases}$

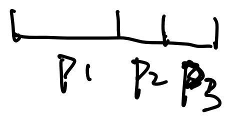
need to prove (associativity)

$([p_1]+[p_2])+[p_3] \;=\; [p_1]+([p_2]+[p_3])$



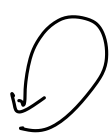
$\neq$

$\simeq$



b) identity $c_{x_0} = t \mapsto x_0$

constant path on x_0

c) inverse  $p \rightsquigarrow p^{-1}$ 

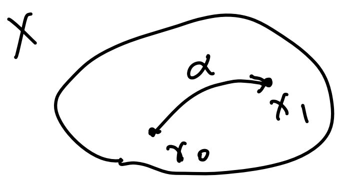
$t \mapsto p(1-t)$

need to show

exercise

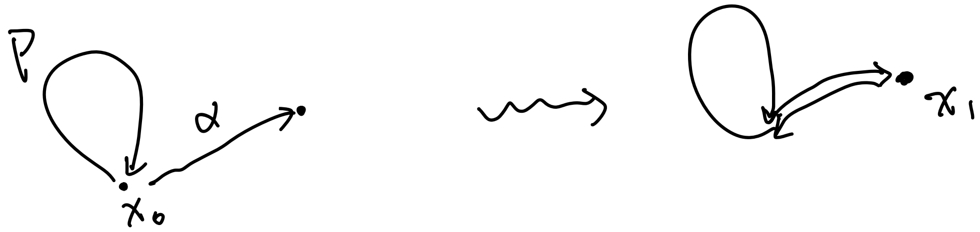
- $[p] + [p]^{-1} = [c_{x_0}]$
- $[c_{x_0}] + [p] = [p]$
- $[p] + [c_{x_0}] = [p]$
- independence of representetives

Base point (in)dependence



suppose $\exists$ path $x_0 \xrightarrow{\alpha} x_1$

$\rightsquigarrow$ α gives an assignment



$p \in \pi_1(X, x_0) \mapsto \alpha p \alpha^{-1} \in \pi_1(X, x_1)$

Claim: this defines a grp homomorphism

$\gamma(\alpha) : \pi_1(X, x_0) \to \pi_1(X, x_1)$

$\gamma(\alpha^{-1}) : \pi_1(X, x_1) \to \pi_1(X, x_0)$

Claim: $\gamma(\alpha)$ inverse to $\gamma(\alpha^{-1})$ $\rightsquigarrow$ $\pi_1(X, x_0) \cong \pi_1(X, x_1)$

Q: $\begin{tikzcd} x_0 \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & x_1 \end{tikzcd}$ is $\gamma(\alpha) \overset{?}{=} \gamma(\beta)$?

A: If $\pi_1(X, x_0)$ abelian, yes. exercise

homotopy invariance

prop $f \simeq g : X \to Y$ $\quad$ $f(x_0) = y_0$, $g(x_0) = y_1$

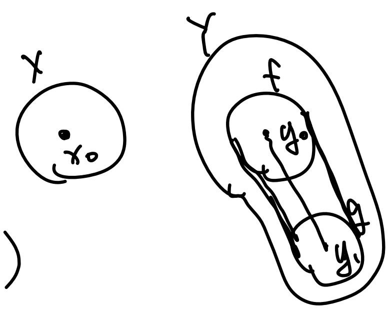
then

$$\begin{tikzcd} & \pi_1(X, x_0) \arrow[dl, "f_*"'] \arrow[dr, "g_*"] & \\ \pi_1(Y, y_0) \arrow[rr, "\gamma(\alpha)"'] & & \pi_1(Y, y_1) \end{tikzcd}$$

is commutative.

Here

- $\alpha = H|_{\{x_0\} \times I} : y_0 \xrightarrow{\text{path}} y_1$



- $f_* :$ $[p] \in \pi_1(X, x_0)$
$$\downarrow$$
$[f \circ p] \in \pi_1(Y, y_0)$

$f : X \to Y$
$\Omega p : S^1 \to X. \quad \in \pi_1(X)$
$f \circ p : S^1 \to X \to Y \in \pi_1(Y)$

similar for g_*.

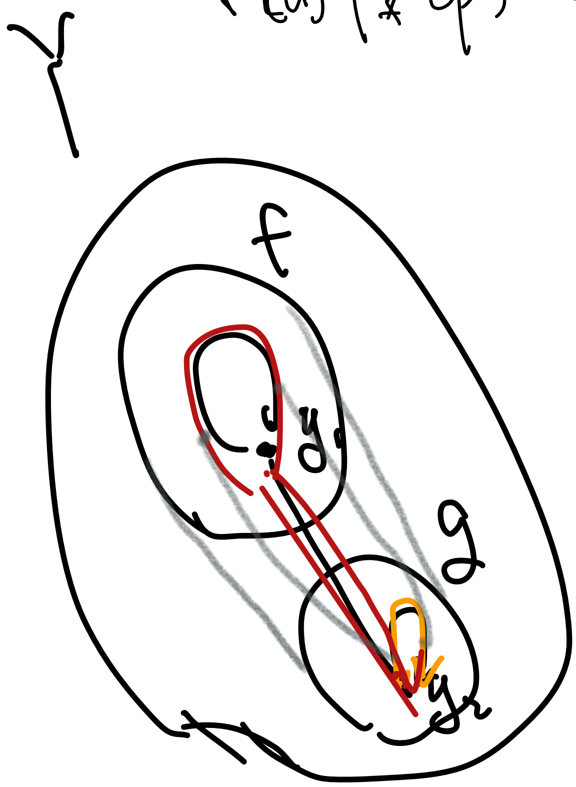
exercise: check well-definedness of f_*, g_*

pf want to show $\tau(\alpha) f_* = g_*$

$p : I \longrightarrow X$ with $p(0) = p(1) = x_0$

$g_*(p) = \underline{g \circ p} : I \xrightarrow{p} X \xrightarrow{g} Y$

$\tau(\alpha) f_*(p) = \tau(\alpha)\, f \circ p = \underline{\alpha\, f \circ p\, \alpha^{-1}}$

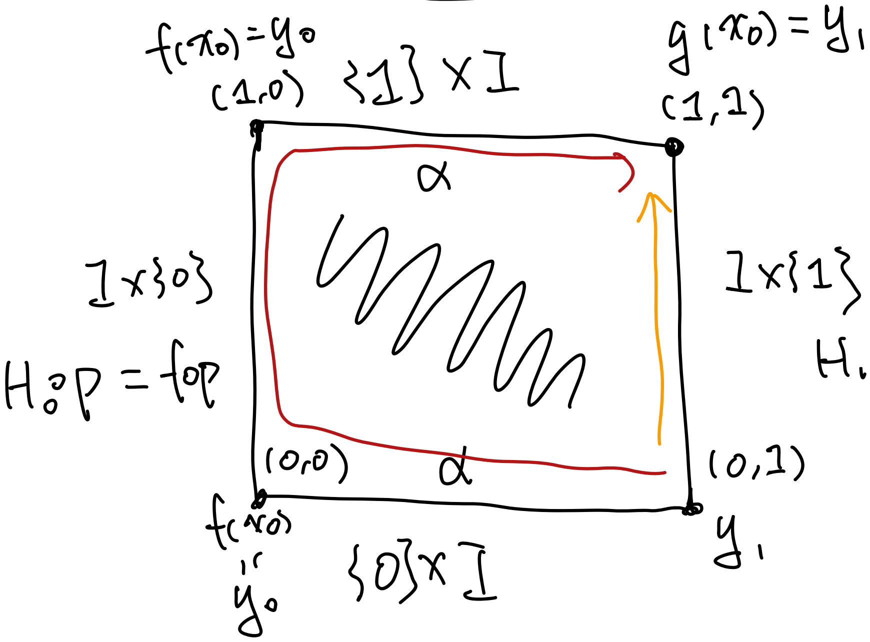


$H : X \times I \longrightarrow Y$

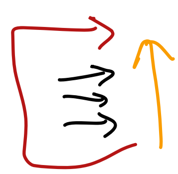
$p : I \longrightarrow X$

$H \circ (p \times I)$:

$$I \times I \xrightarrow{p \times I} X \times I \xrightarrow{H} Y$$



$\exists$ a homotopy



$\rightsquigarrow g \circ p \sim \alpha\, f \circ p\, \alpha^{-1}$

Cor $X \xrightarrow{f} Y$ homotopy equivalence

Then $f_*: \pi_1(X) \longrightarrow \pi_1(Y)$ is isomorphism.

sketch. ① $\exists\, g$ s.t. $g\circ f \simeq id_X$
use $\qquad f\circ g \simeq id_Y$

② $(id_X)_* = id_{\pi_1(X)}$
$(id_Y)_* = id_{\pi_1(Y)}$

Computation

Def. If $X \simeq *$, we say X is contractible.

Ex: $\mathbb{R}^1$

①. Using Cor $\Rightarrow$ $\pi_1(\mathbb{R}) =$ trivial grp.

② $\pi_1(S^1) = \mathbb{Z}$

1) identify S^1 with unit sphere $\subseteq \mathbb{C}$

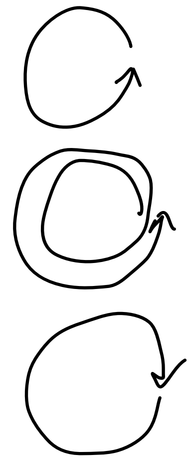
$t \longmapsto e^{2\pi i n t}$

$[0,1] \longrightarrow S^1 \subseteq \mathbb{C}$

$n = 1$

$n = 2$

$n = -1$

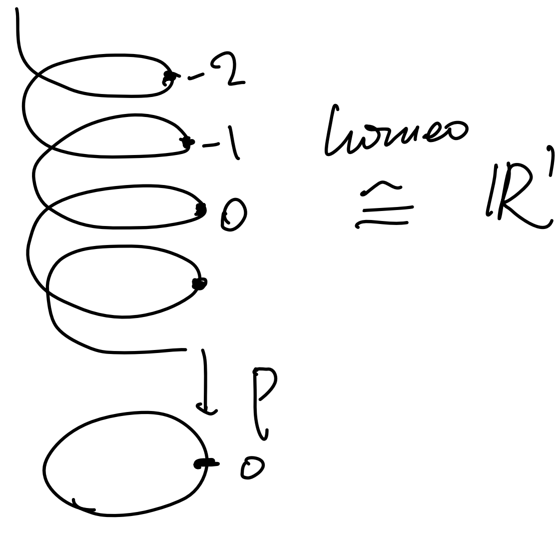


$$i : \mathbb{Z} \longrightarrow \pi_1(S^1)$$
$$n \longmapsto t \mapsto e^{2\pi i n t}$$

Claim : i is a homomorphism

$$j : \pi_1(S^1) \longrightarrow \mathbb{Z}$$

$$I \xrightarrow{\tilde{p}} \mathbb{R}^1, \qquad I \xrightarrow{p} S^1$$



Claim : given $p : I \to S^1$ with $p(0) = p(1) = 0$

$\exists !$ lift $\tilde{p} : I \to \mathbb{R}^1$ with $\tilde{p}(0) = 0$, $\tilde{p}(1) = n$ for some integer n

$j : p \longmapsto \tilde{p}(1)$ $= \deg$

Claim: 1) j is well-defined

2) $j \circ i = id_{\mathbb{Z}} \Rightarrow$ i inj, j surj

3) j is inj $\Rightarrow i$ is bij

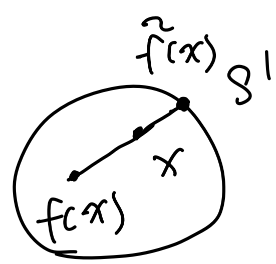
Application

① **Brouwer fixed pt th'm**

Any continuous map $D^2 \to D^2$ has a fixed pt.

Pf suppose $f: D^2 \to D^2$ with no fixed pt.

$\tilde{f} : x \longmapsto$



get $\tilde{f} : D^2 \longrightarrow S^1$ with

$\tilde{f}|_{S^1} = id_{S^1}$

$id_{S^1} = S^1 \xhookrightarrow{\text{inclusion}} D^2 \xrightarrow{\tilde{f}} S^1$

Apply π_1

$$\mathrm{id} = (\mathrm{id}_{S^1})_* = \begin{array}{ccccc} \pi_1(S^1) & \xrightarrow{(\mathrm{inc})_*} & \pi_1(D^2) & \xrightarrow{f_*} & \pi_1(S^1) \\ \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} \end{array}$$

contradiction!

② fundamental thm of alg

$f(x) = x^n + c_1 \cdot x^{n-1} + c_2 x^{n-2} + \cdots + c_n$

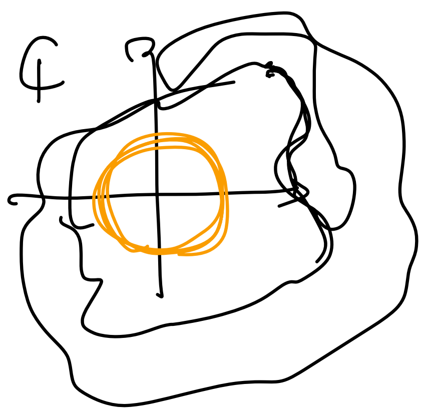
then $f(x)$ has n roots in $\mathbb{C}$.
(consider multiplicity)

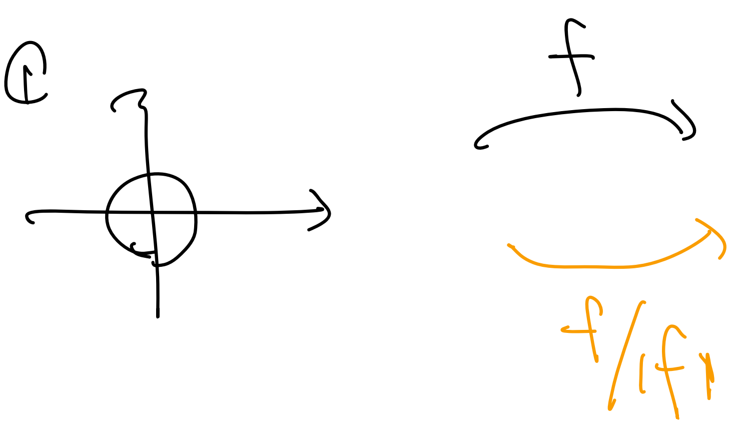
pf only need to show
$f(x)$ has 1 root.

Suppose $f(x)$ doesn't have any roots.

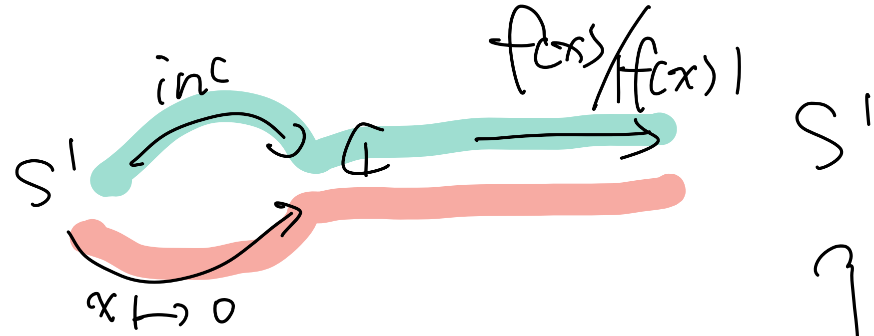
$$S^1 \hookrightarrow \mathbb{C} \xrightarrow{f(x)/|f(x)|} S^1 \subseteq \mathbb{C}$$

$\in \pi_1(S^1) = \mathbb{Z}$

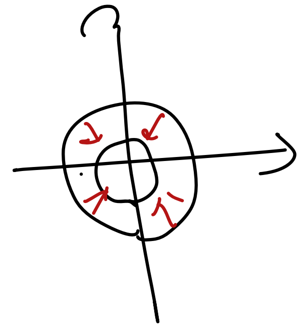


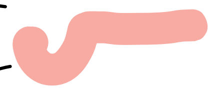


①.



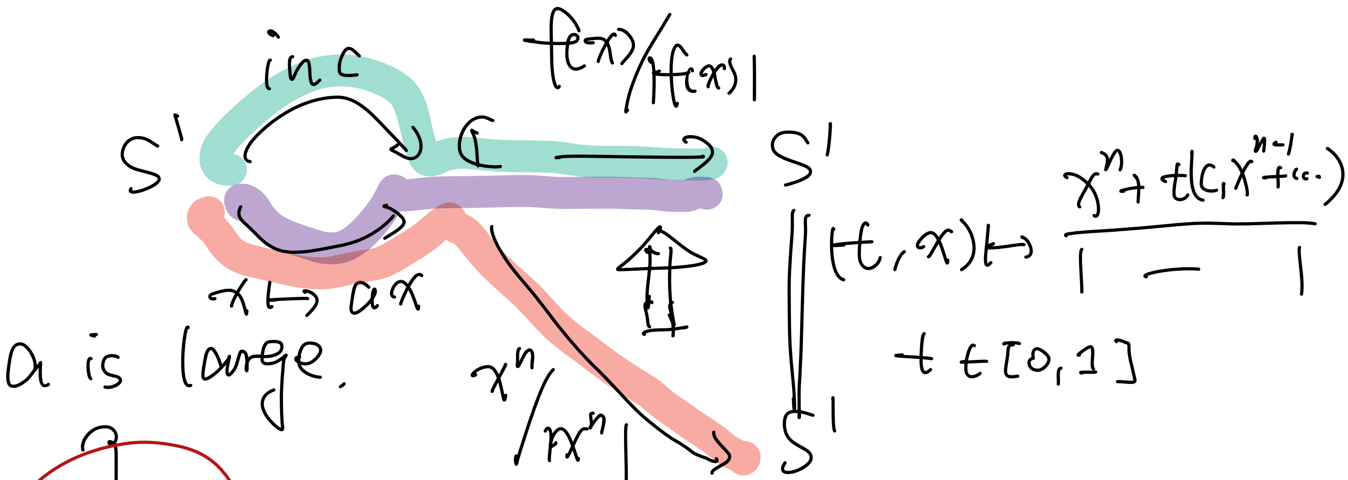
$(\mathrm{inc}) \sim (x \mapsto 0)$

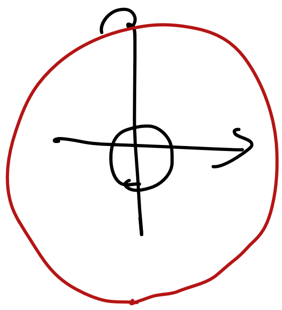


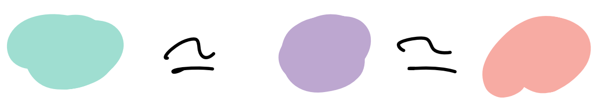
$\Rightarrow$ [] = [] $\in \pi_1(S^1)$

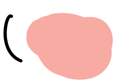
$\deg = 0$

②







deg() $= n$

② contradicts ①

Lemma

$$\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$$

sketch

$$p : I \to X \times Y$$

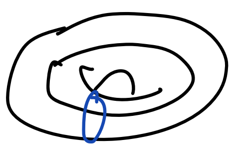
$$p_X : I \to X \times Y \to X$$

$$p_Y : I \to X \times Y \to Y$$

$$[p] \mapsto ([p_X], [p_Y])$$

Exercise show ↑ is grp isomorphism.

Ex $\pi_1(T^2) = \pi_1(S^1 \times S^1) = \mathbb{Z} \times \mathbb{Z}$



Q: $X = U \cup V$

know $\pi_1(U)$, $\pi_1(V)$, $\pi_1(U \cap V)$ & induced maps

How to compute $\pi_1(X)$?

A: van Kampen thm

Some category theory

Def (Category).

a category $\mathcal{C}$ is the following data

① obj : $\mathrm{ob}\,\mathcal{C}$.

② $a, b \in \mathrm{ob}\,\mathcal{C}$, a set of morph $\mathcal{C}(a,b)$

with $id_a \in \mathcal{C}(a,a)$

③ composition that is associative

$$\mathcal{C}(a,b) \times \mathcal{C}(b,c) \xrightarrow{\circ} \mathcal{C}(a,c)$$

associative:

$$(f \circ g) \circ h = f \circ (g \circ h)$$

Ex : ⓪ Set , Top , Top$_*$ (pointed top spaces) , Grp

① G : group

$\mathcal{G}$: the cat associated to G.

obj : $\bullet \circlearrowleft g$

morph : $\mathcal{G}(\bullet, \bullet) = G$

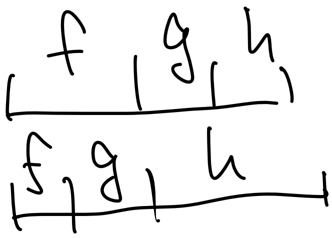
C_2 :

② fundamental groupoid for X

$\Pi(X)$ obj: points in X

morph: $\Pi(X)(x, y)$

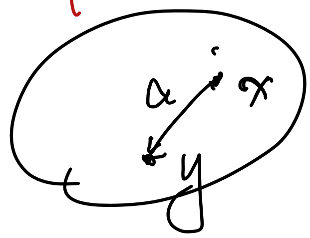
$= \{\text{paths } x \to y\} / \text{homotopy}$



necessary for composition

$x \xrightarrow{\alpha} y$

$y \xrightarrow{\alpha^{-1}} x$



groupoid: each morphism has an inverse.

Functors

Def: a functor $F: \mathcal{C} \to \mathcal{D}$ is

① a map $F: \mathrm{ob}\,\mathcal{C} \to \mathrm{ob}\,\mathcal{D}$

② $F: \mathcal{C}(a, b) \to \mathcal{D}(F(a), F(b))$

$F(id_a) = id_{F(a)}$

respect composition.

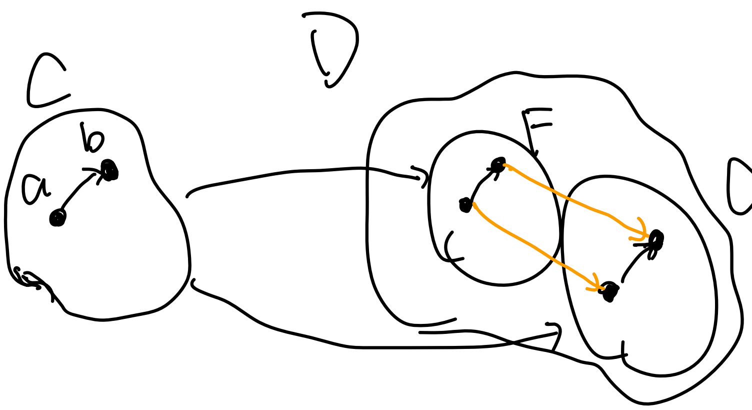
$F(f) \circ F(g) = F(f \circ g)$.

Def (natural transformation). $F, G : \mathcal{C} \to \mathcal{D}$

a natural transformation $\eta : F \Rightarrow G$ is the following:

for $\forall a \in \mathcal{C}$, there is a morph $\eta_a : F(a) \longrightarrow G(a) \in \mathcal{D}$, s.t.

$$\begin{tikzcd} F(a) \arrow[r, "\eta_a"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\circlearrowright"] & G(a) \arrow[d, "G(f)"] \\ F(b) \arrow[r, "\eta_b"] & G(b) \end{tikzcd}$$



I: $0 \longrightarrow 1$

claim: a natural tranf $\underline{\underline{\text{same data}}}$ $H : \mathcal{C} \times I \longrightarrow \mathcal{D}$

s.t. $H|_0 = F$

$H|_1 = G$

Def (natural isomorphism).

natural transformation + each η_a is an isomorphism/equivalence.

a morphism is an iso/eq if it is invertible.

Space	cat.
conti map	functor
homotopy	natural transformation (isomorphism)

$X \xrightarrow{f} Y$ homotopy eq if.

$\exists g : Y \to X$ s.t. $f \circ g \simeq id$
$g \circ f \simeq id$

$\mathcal{C} \xrightarrow{F} \mathcal{D}$ equivalence if

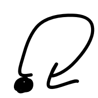
$\exists G : \mathcal{D} \to \mathcal{C}$ s.t.

$G \circ F \cong id$
$F \circ G \cong id$

natural isomorphism

Prop X is path connected.
Then $\Pi(X)$ & $\pi_1(X)$ viewed as a cat
are equivalent.

finite Set	$\{1,2,3\}$ $\{a,b,c\}$	taking skeleton	$\Pi(X)$
$\downarrow$	$\downarrow$		$\downarrow$
$\mathbb{Z}$	3		$\pi_1(X)$



---

colimit.

$\mathcal{D}$: (small) category

Ex $\mathcal{D}$: 1) $\bullet \quad \bullet$

2)
$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd}$$

3) $\to \bullet \to \bullet \to \bullet \to$

$F : \mathcal{D} \to \mathcal{C}$
is a diagram in $\mathcal{C}$.

① $c_1 \quad c_2$

②
$$\begin{tikzcd} c_1 \arrow[r] \arrow[d] & c_2 \\ c_3 \end{tikzcd}$$

③ $\to c_1 \to c_2 \to \cdots$

Def (colimit)

colim F is ① $c \in \mathcal{C}$

② $F \Rightarrow \mathrm{Const}_c$ $\begin{pmatrix} \mathcal{D} \to \mathcal{C} \\ d \mapsto c \end{pmatrix}$

satisfying for $\forall\, F \Rightarrow \mathrm{const}_a$

$\exists!\ c \to a$ s.t.

(universal property)

$$\begin{tikzcd} & F \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\ \mathrm{const}_c \arrow[rr, Rightarrow] \arrow[urr, phantom, "\circlearrowleft"] & & \mathrm{const}\ a \end{tikzcd}$$

Ex

2) : colim
$$\begin{tikzcd} c_1 \arrow[r] \arrow[d] & c_2 \\ c_3 \end{tikzcd}$$

① a obj $c \in \mathcal{C}$.

②
$$\begin{tikzcd} c_1 \arrow[r] \arrow[d] \arrow[dr] & c_2 \arrow[d] \\ c_3 \arrow[r] & c \end{tikzcd}$$

satisfying

for $\forall$ a with

$$\begin{tikzcd} c_1 \arrow[r] \arrow[d] \arrow[ddr] \arrow[drr] & c_2 \arrow[dd] \arrow[dr] & \\ c_3 \arrow[dr] \arrow[rr] & & a \\ & c \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

(more on this example) $\mathcal{C}$ is Top

$$\begin{tikzcd} X_1 \arrow[r, hook] \arrow[d] & X_2 \arrow[d] \arrow[dr] & \\ X_3 \arrow[r] \arrow[rr, bend right] & X \arrow[r, dashed] & A \end{tikzcd}$$

$X = X_2 \amalg_{X_1} X_3$

glue X_2 & X_3 along X_1

Thm (van Kampen)

- X path connected, choose base pt x_0
- $\mathcal{O}$: a cover of X, such that $\mathcal{O}$ is closed under taking intersection of finitely many elements in $\mathcal{O}$ meaning

i.e. $\mathcal{O} = \{U_i\}$, $\bigcap_{i \in I} U_i \in \mathcal{O} \quad \forall |I| < \infty$

- X & all U_i are path connected
- view $\mathcal{O}$ as a category in this way

ob $\mathcal{O}$: $\{U \in \mathcal{O}\}$

morph : $\mathcal{O}(U, U') = \begin{cases} \to & U \hookrightarrow U' \\ \emptyset & U \not\hookrightarrow U' \end{cases}$

$\mathcal{O}$: $U_1, U_2, U_1 \cap U_2$ this is an example

- consider the functor $F : \mathcal{O} \to$ groupoids.
$U_i \to \Pi(U_i)$

Then $\Pi(X) = \operatorname{colim} F$

(or written as $\operatorname{colim}_{\mathcal{O}} \Pi(U_i)$)

this version of van Kampen

$\Rightarrow$ translate Π to π, using $\Pi(X)_x = \pi_1(X)$

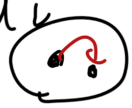
$\Rightarrow$ write out formulas for colim in grp.

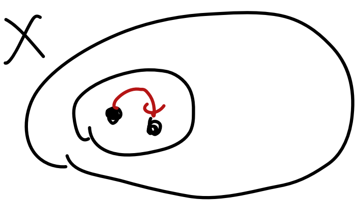
$\Rightarrow$ Hatcher wikipedia version

in the textbook Chap 2, p18 ①

Think about what colimits are in cat of grps. ②

Pf: to verify the universal property.

U_i 

X 

$$\begin{tikzcd} \Pi(U_i) \arrow[d, "i"'] \arrow[dr, "F_{U_i}"] & \\ \Pi(X) \arrow[r, dashed, "F'"'] & A \end{tikzcd}$$

want

A ← a grpoid.

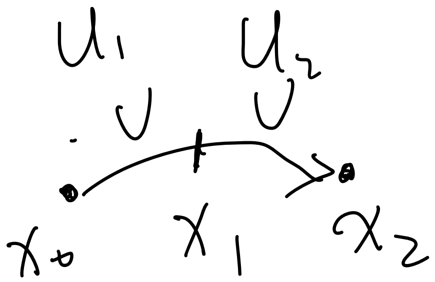
on obj $x \in X$ also in U

$$\begin{tikzcd} x \arrow[d, "i"'] \arrow[dr, bend left, "F_U"] & \\ x \arrow[r, dashed, "F'"'] & F(x) \end{tikzcd}$$

$F'(x)$ determined by $\Pi(U) \to A$.

on morph. ① If a path in X is entirely in some U ✓

② Otherwise, subdivide.



By ① we know how

$x_0 \to x_1$ &

$x_1 \to x_2$

are mapped to A.

Define $F'(x_0 \to x_2)$

$$:= F_{U_2}(x_1 \to x_2) \circ F_{U_1}(x_0 \to x_1)$$

Check ① everything well defined

② uniqueness of F is clear.