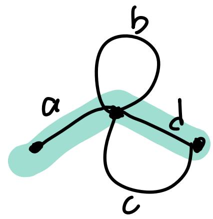
Graph   vertices X^0 $\vdots$

edges.   a set J, $i_j : S^0 \to X^0,\ j \in J$

$$X^0 \amalg I \times J \,/\sim\ = \operatorname{colim}\left(\begin{array}{ccc} S^0 \times J & \to & X^0 \\ \downarrow & & \\ I \times J & & \end{array}\right)$$



$X^0 = {}_1\bullet \quad \overset{2}{\bullet} \quad \bullet_3$

$J = \{a, b, c, d\}$

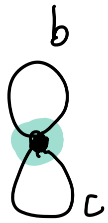
$a: S^0 \to {}_1\bullet \quad \bullet_2$

$b: S^0 \to \overset{\bullet}{2} \quad \overset{\bullet}{2}$

$c: S^0 \to \overset{\bullet}{2} \quad \overset{\bullet}{3}$

$d: S^0 \to \overset{\bullet}{2} \quad \overset{\bullet}{3}$

$G/T \rightsquigarrow$



tree : no closed reduced path

maximal tree $\Leftrightarrow$ contains all vertices
& is a tree

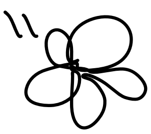
fact : a tree $\simeq *$

prop : G graph, $T \subset G$ maximal tree

$$\begin{array}{ccc} T & \xrightarrow{\simeq} & * \\ \downarrow & & \downarrow \\ G & \xrightarrow{\star} & G/T \end{array}$$

$\star$ is a homotopy equivalence.

G/T =



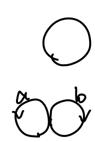
$\Rightarrow \pi_1(G) \simeq \pi_1(G/T)$

$= \pi_1(\bigvee_{\alpha \in A} S^1)$

$=$ free grp.

$\Rightarrow \pi_1(G)$ determined by #vertices $-$ #edges $+1$
when finite

π_1



$\mathbb{Z}$

aba^{-1}

cofibration

Def $A \xrightarrow{i} X$ is cofibration if

$\forall$ commutative

$$\begin{tikzcd}
A \arrow[rr, "id \times \{0\}"] \arrow[d, "i"'] & & A \times I \arrow[dl, "H"'] \arrow[d, "i \times id"] \\
& Y & \\
X \arrow[ur, "f"] \arrow[rr, "id \times \{0\}"'] & & X \times I \arrow[ul, dashed, "\tilde{H}"']
\end{tikzcd}$$

$\exists$ $\tilde{H}$ s.t. everything commutes.

- H : a homotopy $f' \sim g' : A \to Y$
- $\begin{tikzcd} A \arrow[r, "f'"] \arrow[d, "i"'] & Y \\ X \arrow[ur, "f"'] & \end{tikzcd}$ is commutative
- $\exists \tilde{H}$: a homotopy $f \sim$ sth & $\tilde{H}$ extend H

$$Map(X \times Y, Z) = Map(X, Map(Y, Z))$$

$$(x, y) \mapsto z \quad \longmapsto \quad x \longmapsto (y \mapsto z)$$

$$
\begin{tikzcd}
A \arrow[r, "\mathrm{id}\times\{0\}"] \arrow[d, hook, "i"'] & A\times I \arrow[d, "i\times \mathrm{id}"] \arrow[dl, "H"'] \\
X \arrow[r, "\mathrm{id}\times\{0\}"'] \arrow[ur, phantom] & X\times I \arrow[ul, phantom] \\
\end{tikzcd}
\qquad
\begin{tikzcd}
 & Y & \\
X \arrow[ur, "f"] & & X\times I \arrow[ul, dotted, "\tilde{H}"']
\end{tikzcd}
$$

$$\Longleftrightarrow$$

$$
\begin{tikzcd}
A \arrow[r, "H"] \arrow[d, hook, "i"'] & \mathrm{Map}(I,Y) \arrow[d, "ev_0"] \\
X \arrow[r, "f"'] \arrow[ur, dotted, "?"] & Y
\end{tikzcd}
$$

$$\Updownarrow$$

$$
\begin{tikzcd}
\{0\} \arrow[r, "f"] \arrow[d, hook] & \mathrm{Map}(X,Y) \arrow[d, "i\circ(-)"] \\
I \arrow[r, "H"'] \arrow[ur, dotted, "\tilde{H}"] & \mathrm{Map}(A,Y)
\end{tikzcd}
$$

$A \xrightarrow{i} X \to Y$

Homotopy extension property

- Fibration (Dual version)
  (invert all arrows)

<u>Def</u>. $E \xrightarrow{p} B$ is a fibration if

cof

$$
\begin{tikzcd}
\{0\} \arrow[r, "f"] \arrow[d, hook] & \mathrm{Map}(X,Y) \arrow[d, "i\circ(-)"] \\
I \arrow[r, "H"'] \arrow[ur, dotted, "\tilde{H}"] & \mathrm{Map}(A,Y)
\end{tikzcd}
$$

$\forall$ comm.

fib

$$
\begin{tikzcd}
\{0\} \arrow[r, "f"] \arrow[d, hook] & \mathrm{Map}(Y,E) \arrow[d] \\
I \arrow[r, "H"'] \arrow[ur, dotted, "\exists \tilde{H}"] & \mathrm{Map}(Y,B)
\end{tikzcd}
$$

$\exists$ a lift $\tilde{H}$.

$Y \to E \xrightarrow{p} B \rightsquigarrow \mathrm{Map}(Y,E) \to \mathrm{Map}(Y,B)$

When $Y = *$ $\rightsquigarrow$ path lifting property

<u>prop</u>: $A \xrightarrow{i} X$ is cofibration,

$A \xrightarrow{f} B$ is any map.

Then $B \to \operatorname{colim}\left(\begin{array}{ccc} A & \to & B \\ \downarrow & & \\ X & & \end{array}\right)$ is also a cofibration.

$\qquad\qquad\qquad\qquad \cong X \underset{A}{\cup} B$

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ X \arrow[r] \arrow[ur, phantom, "\ulcorner"] & X \underset{A}{\cup} B \end{tikzcd}$$

<u>pf</u>.

$$\begin{tikzcd} ① \ \mathrm{Map}(X,Y) \arrow[d] & \mathrm{Map}(X \underset{A}{\cup} B, Y) \ ③ \arrow[l] \arrow[d] & \{0\} \arrow[l, hook'] \arrow[d, hook] \\ ② \ \mathrm{Map}(A,Y) & \mathrm{Map}(B,Y) \ ④ \arrow[l] & I \arrow[l, hook'] \arrow[u, dashed] \end{tikzcd}$$

①. $\exists$ lift $I \to \mathrm{Map}(X,Y)$.

② $\square$ is a <u>pull back</u> diagram. dual to push out

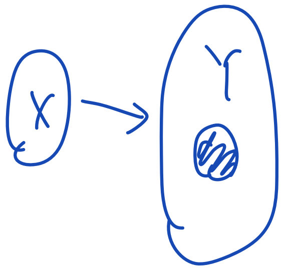
$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd} \qquad \bullet = \lim\left(\begin{array}{cc} & \downarrow \\ \to & \end{array} \right)$$

$$\begin{tikzcd} & ③ \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & ① \arrow[d] \\ I \arrow[ur, dashed] \arrow[urr] \arrow[r] & ② \arrow[r] & ④ \end{tikzcd}$$

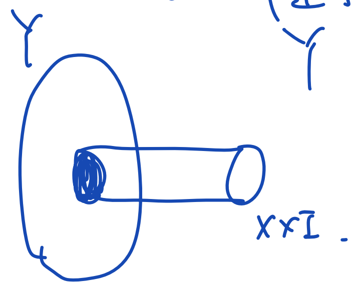
mapping cylinder

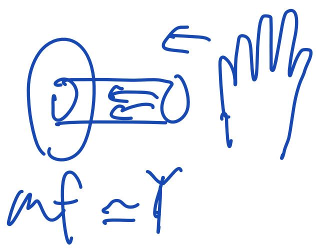
$f: X \longrightarrow Y$, $\quad Mf = Y \amalg (X \times I) \big/ _{f(x) \sim (x,0)}$

$$= \mathrm{Colim}\left(\begin{tikzcd} X \arrow[r, "x \mapsto (x,0)"] \arrow[d, "f"'] & X \times I \\ Y & \end{tikzcd}\right).$$



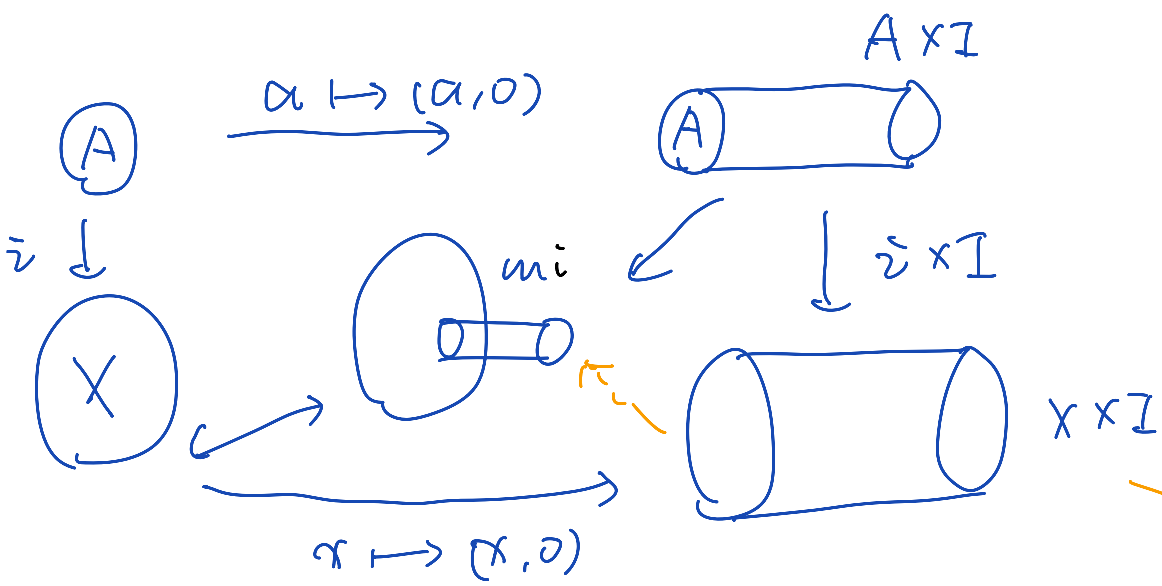
$Mf =$





<u>prop</u> : can replace HEP for any Y

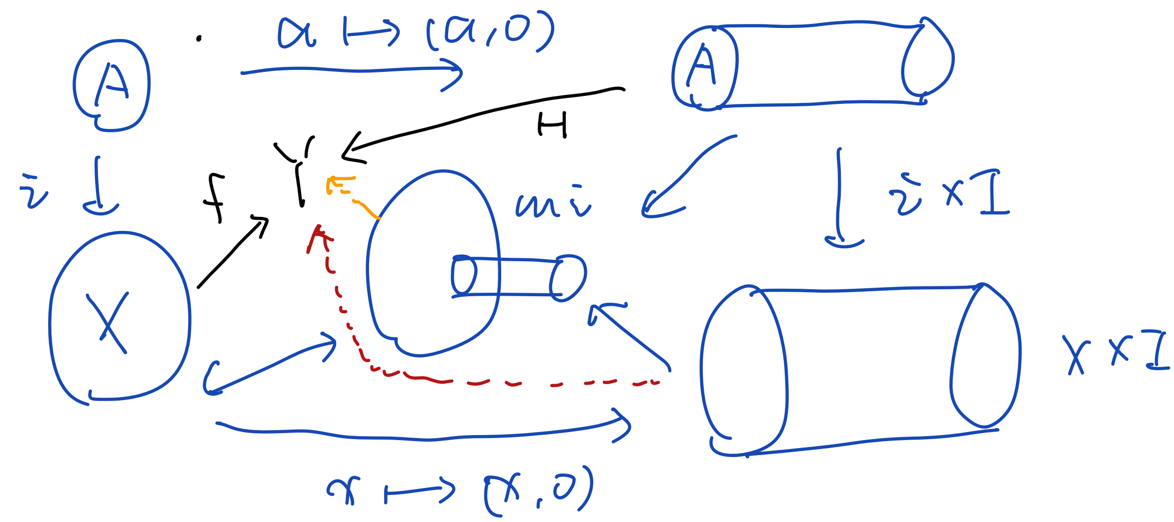
by HEP for Mi with



<u>Pf</u> want show if HEP is satisfied for

then

want to show $\exists$

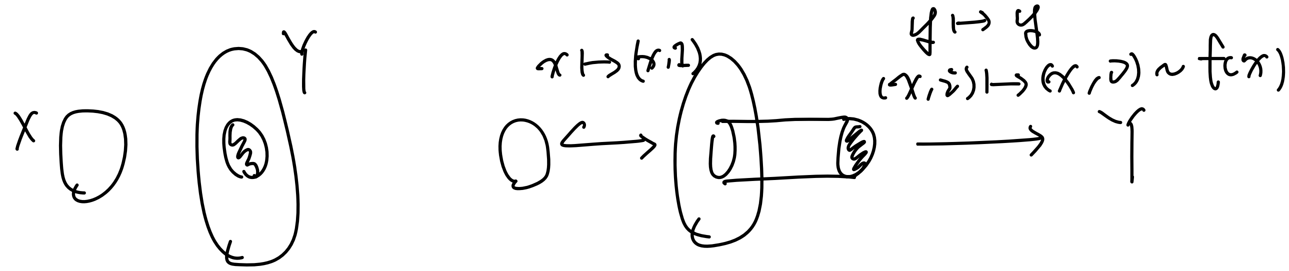


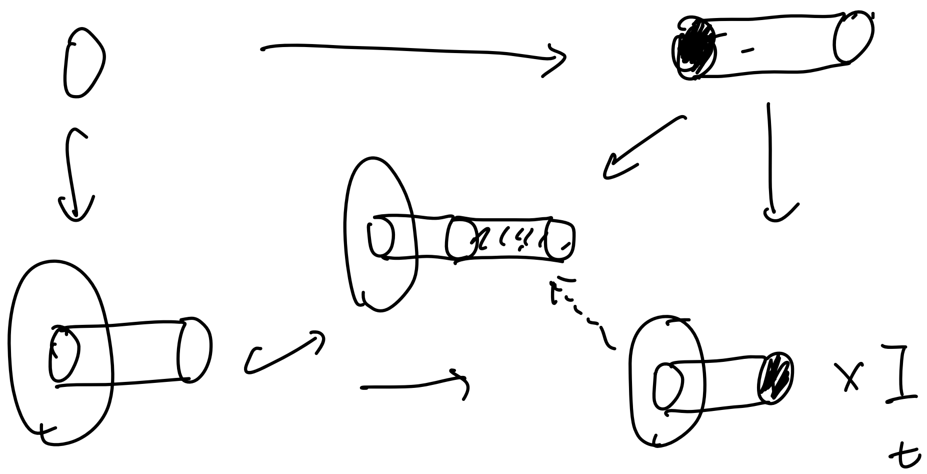
<u>prop</u>. $f: X \to Y$ factors as

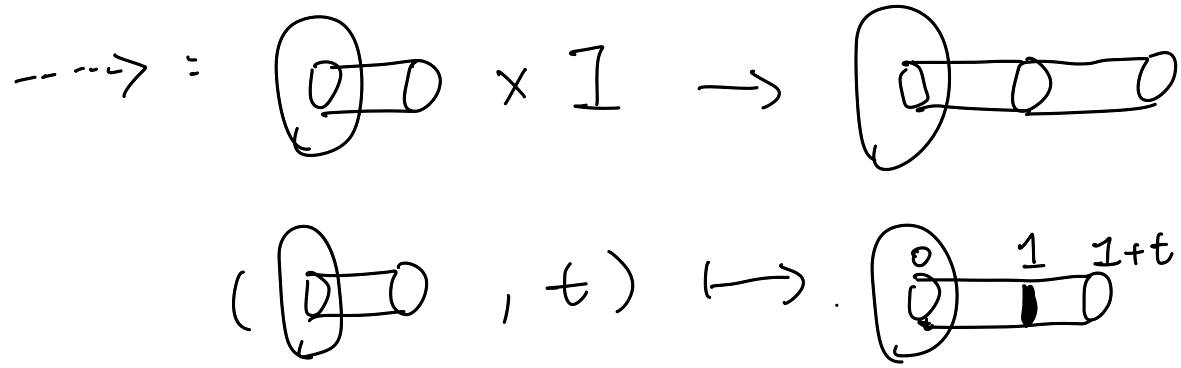
$$X \underset{①}{\hookrightarrow} M_f \underset{②}{\xrightarrow{\simeq}} Y$$

& ① is a cofibration

② is homotopy equivalence.

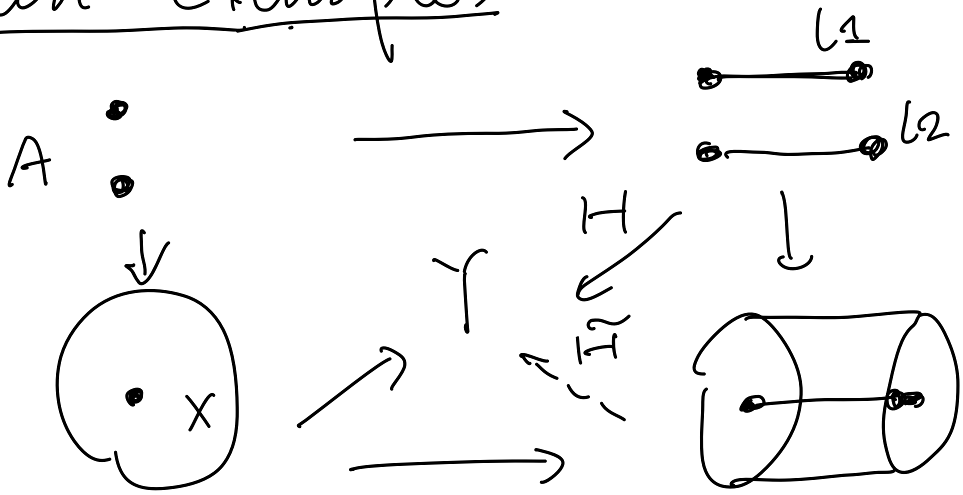






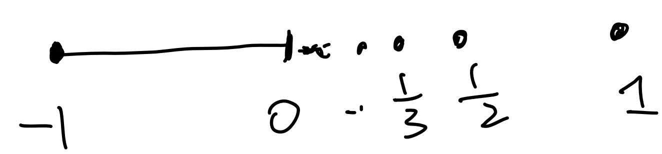
All have dual notions.

<u>non-examples</u>

① 

$\nexists \tilde{H}$ if.

$H|_{i_1} \neq H|_{i_2}$

② 

$A = [-1, 0]$

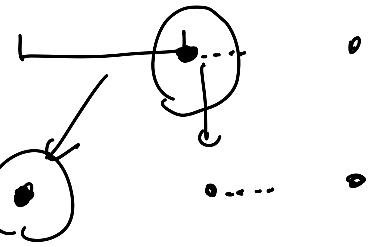
$X = A \cup \{\frac{1}{n}, n \in \mathbb{Z}_+\}$

$A \hookrightarrow X$ is not cof.

$$\begin{tikzcd}
A \arrow[rr] \arrow[dd, hook] & & A \times I \arrow[dl, "H"'] \arrow[dd] \\
& X & \\
X \arrow[ur, "id"] \arrow[rr] & & X \times I \arrow[ul, dashed]
\end{tikzcd}$$

H: shrink A from $[-1, 0]$ to $\{-1\}$

$$\tilde{H}_1 : \quad X \longrightarrow X$$
$$a \longmapsto -1$$
$$\frac{1}{n} \longmapsto \frac{1}{n}$$



- $\tilde{H}$ needs to be an ext of H.

$\tilde{H}|_A \rightsquigarrow \quad \rightsquigarrow \{-1\}$

- $X \longrightarrow [-1, 1]$

$\exists \ \tilde{H}$ = scale everything to -1

$\tilde{H}|_{\{\frac{1}{2}\}} : I \longrightarrow X$

start with $\frac{1}{2}$

When a map is a cof.

<u>Def</u>. ① (X, A) is a neighbourhood deformation retract pair (NDR)

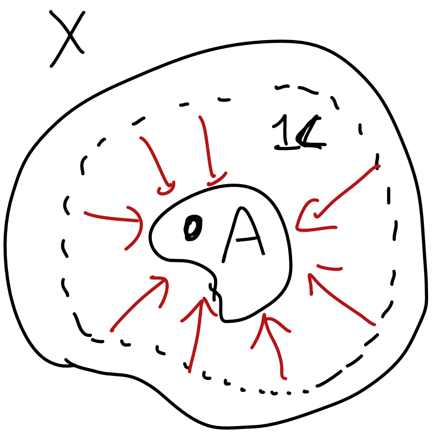
if $\quad \exists\, u : X \to I$

s.t. $u^{-1}(0) = A$

$h : X \times I \to X$ s.t.

$h|_A = id_A$, &

$\mathrm{Im}\; h|_{(u^{-1}[0,1),1)} = A$.

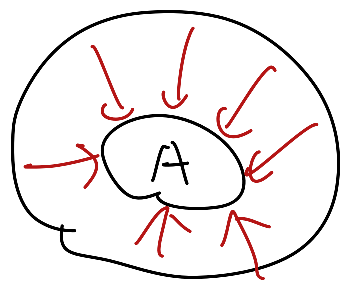


$U = u^{-1}([0, 1))$

② (X, A) is a deformation retract pair (DR)

if NDR pair

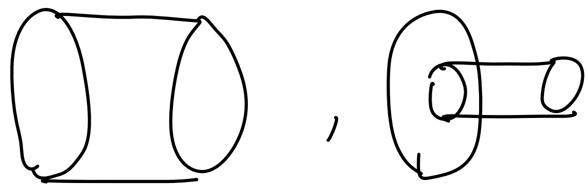
$+\ \mathrm{Im}\; h|_{X \times \{1\}} = A$



Th'm The following are equivalent:

1). $A \hookrightarrow X$ is cofibration

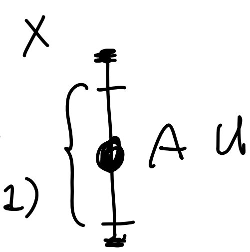
2) (X, A) is a NDR pair

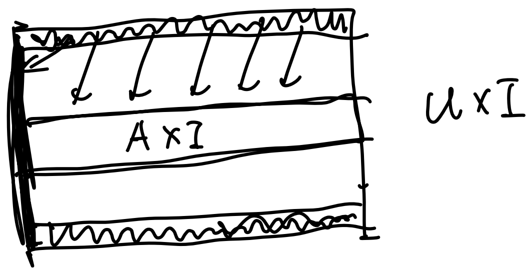
3) () is DR pair

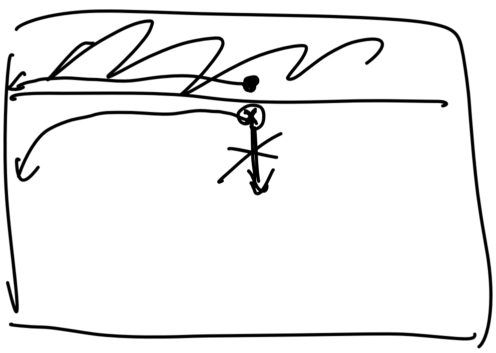
$X \times I$ $\quad$ M_i

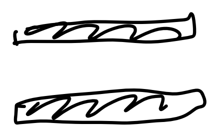
Pf : 1) <=> 3) ✓

2) <=> 3)

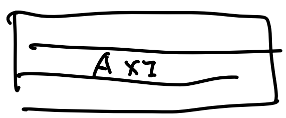
2) => 3) : $u^{-1}[0,1)$ 

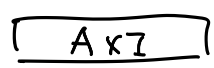




①. Inside 

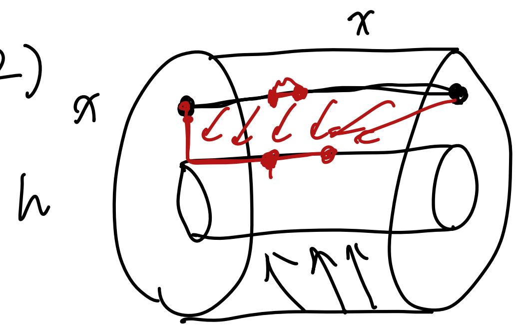
shrink along $I \to \{0\}$

② Inside 

shink to 

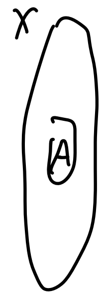
in a compatible way.

3) ⟹ 2)



DR.

then



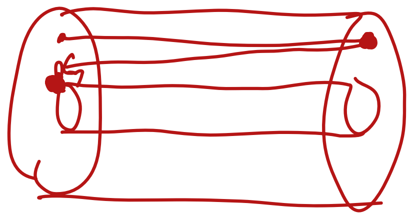
NDR.

$\pi_I : X \times I \to I$

$$u(x) = \sup(t - \pi_I h_1(x,t))$$

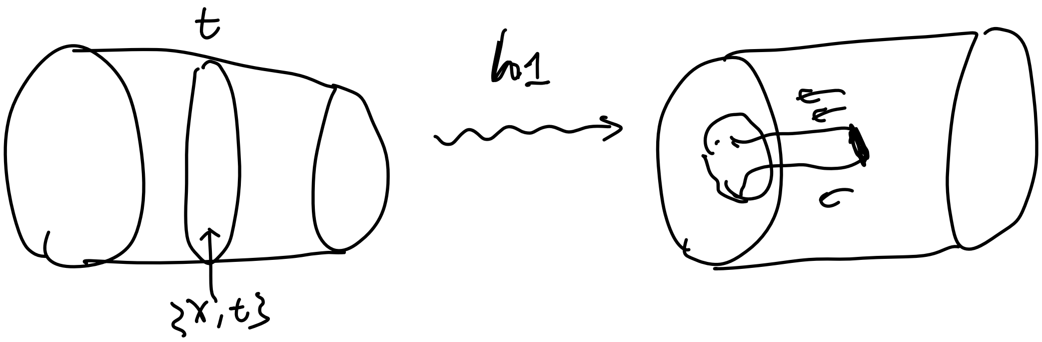
$u^{-1}(0) = A$

$u^{-1}(1) =$



$$H(x,t) := \pi_X(h_1(x,t))$$

$\pi_X : X \times I \to X$



---

prop

$$\begin{tikzcd} & A \arrow[dl, "i"'] \arrow[dr, "j"] & \\ X \arrow[rr, "f"] & & Y \end{tikzcd}$$

i, j are cofibrations

f is a homotopy equivalence

Then f is a homotopy equivalence under A.

rel A:

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d] \arrow[r, bend left, "\tilde{H}"] & A \arrow[d] \\ X \arrow[r, "id"] \arrow[r, bend right, "g\circ f"'] & X \end{tikzcd}$$

$\tilde{H} : \tilde{H}_t = id_A$

H extends $\tilde{H}$

⟺

$\exists\, g : Y \to X$

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ X \arrow[rr, "f"] & & Y \arrow[ll, bend left, "g"] \end{tikzcd}$$

s.t.

$g\circ f \simeq id_X$ rel A

$f\circ g \simeq id_Y$ rel A

$H : g\circ f \simeq id_X$

$H_t|A = id_A$

proof

$$\begin{tikzcd} & A \arrow[dl, "i"'] \arrow[d, "j"] \arrow[dr, bend left, "i"] & \\ X \arrow[r, "f"'] & Y \arrow[r, "g"] \arrow[r, bend right, "g'"'] & X \end{tikzcd}$$

① $\exists g$ s.t. $g \circ f \simeq id$, $f \circ g \simeq id$

② use j is a cofibration to replace g with g' s.t. $g' \simeq g$ & $g' \circ j = i$

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr] & A \times I \arrow[d] \arrow[dl, "gj \simeq i"'] \\ Y \arrow[ur, phantom, ""] \arrow[r] \arrow[ru, phantom, ""] & Y \times I \arrow[ul, phantom, ""] \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[rr] \arrow[d] \arrow[dr] & & A \times I \arrow[dl, "gj \simeq i"'] \arrow[d] \\ Y \arrow[r, "g"] \arrow[rr, bend right] & X & Y \times I \arrow[l, dashed] \end{tikzcd}$$

$\rightsquigarrow \exists \tilde{H} : Y \times I \to X$ extends $gj \simeq i$

$\tilde{H}_1 j = i \rightsquigarrow$ let $g' := \tilde{H}_1$

③
$$\begin{tikzcd} A \arrow[rr] \arrow[d, "i"'] & & A \times I \arrow[dl, "H_t : A \xrightarrow{i} X"'] \arrow[d] \\ X \arrow[r, "g'f"] \arrow[rr, bend right] & X & X \times I \arrow[l, dashed, "H'"'] \end{tikzcd}$$

$H'_t|_A = H_t = i : A \to X$

$\rightsquigarrow \quad g' \circ f \overset{H'}{\simeq} id_X$

H' is relative to A.

prop

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[d, "i"'] & B \arrow[d, "j"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

i, j cofibrations

d, f are h.e.

Then (d, f) is a homotopy equivalence of pairs $(X, A) \to (Y, B)$

$\exists$ e h. inverse of d

g h. inverse of f.

$H : ed \simeq id_A$

$\tilde{H} : gf \simeq id_X$

s.t.
$$\begin{tikzcd} A \arrow[r, "d"] \arrow[d, "i"'] \arrow[rr, bend left, "id"] \arrow[rr, phantom, bend left=20, "H"] & B \arrow[r, "e"] \arrow[d, "j"] & A \arrow[d, "i"] \\ X \arrow[r, "f"'] \arrow[rr, bend right, "id"'] \arrow[rr, phantom, bend right=20, "\tilde{H}"'] & Y \arrow[r, "g"'] & X \end{tikzcd}$$

$\tilde{H}$ extends H

$\boxed{\text{Pf}}$

①

$$\begin{tikzcd} A \arrow[r, "d"', "\simeq"] \arrow[d, "i"'] \arrow[rr, bend left, "id"] & B \arrow[r, "e"] \arrow[d, "j"] & A \arrow[d, "i"] \\ X \arrow[r, "f"'] \arrow[rr, bend right, "id"', "\simeq"] & Y \arrow[r, "g"'] & X \end{tikzcd}$$

find h.inverse e, g.

② make $\square$ commutative

use HEP ~~for~~ j ⤳ replace g with g' s.t. $g' \simeq g$ & g' makes $\boxed{2}$

③

$$\begin{tikzcd} A \arrow[r, "ed"] \arrow[d] \arrow[r, bend left=60, "id_A"] & A \arrow[d] \\ X \arrow[r, "fg'"] \arrow[r, bend right=60, "id"'] & X \end{tikzcd}$$

extend $ed \simeq id_A$ with starting point $= fg'$

⤳ $\tilde{H} : X \times I \to X$

s.t. $\tilde{H}_0 = fg'$

$i \circ \tilde{H}_1 = i$

④

$$\begin{tikzcd} A \arrow[r, "ed"] \arrow[d, "i"'] \arrow[r, bend left=60, "id_A"] & A \arrow[r] \arrow[d, "i"] & A \arrow[d] \\ X \arrow[r, "fg'"] \arrow[r, bend right=60, "\tilde{H}_1"'] \arrow[rr, bend right=80, "id"'] & X \arrow[r, "h"] & X \end{tikzcd}$$

h: h.inverse of $\tilde{H}_1$

$h \circ \tilde{H}_1 \simeq id$ rel A.

Fibration

Def $p: E \to B$ is a fibration if it satisfy covering homotopy property (CHP):

$$\begin{tikzcd}
\{0\} \arrow[r] \arrow[d] & \mathrm{Map}(Y,E) \arrow[d, "\circ p"] \\
I \arrow[r] \arrow[ur, dashed] & \mathrm{Map}(Y,B)
\end{tikzcd}
\iff
\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "\mathrm{id}\times\{0\}"'] & E \arrow[d, "p"] \\
Y\times I \arrow[r, "H"'] \arrow[ur, dashed, "\tilde{H}"] & B
\end{tikzcd}$$

for any commutative diagram
$\exists\ \tilde{H}$ to make the diag commutative.

prop: $p: E \to B$ fibration
$A \xrightarrow{f} B$ any map

$$\begin{tikzcd}
E\times_B A \arrow[r] \arrow[d] & E \arrow[d] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

Then $E\times_A B \to A$ is fibration.

$E\times_B A$ is pullback
$= \lim\left(\begin{tikzcd} & E \arrow[d, "p"] \\ A \arrow[r, "f"'] & B \end{tikzcd}\right)$

$f: X \to Y$

- Mf. mapping cylinder $\quad Mf = \mathrm{colim}\left(\begin{tikzcd} X \arrow[r] \arrow[d] & X\times I \\ Y & \end{tikzcd}\right)$
- Nf : mapping path space. $\quad Nf = \lim\left(\begin{tikzcd} & \mathrm{Map}(I,Y) \arrow[d, "ev_0"] \\ X \arrow[r] & Y \end{tikzcd}\right)$

$$Nf = \{(x, \gamma) \mid f(x) = \gamma(0)\}$$

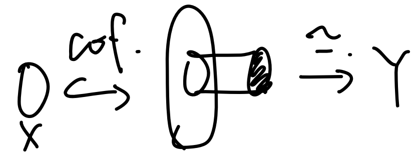
wnt ① universal testing diagram for CHP

② replace any map $X \xrightarrow{f} Y$ with a composite of a h.e. & a fibration

exercise.

①: write down the diagram

②: $X \to Nf \to Y$



exercise   what are these maps?
prove one is h.e.
one is a fibration.

---

prop.
$$\begin{tikzcd} D \arrow[rr, "f"] \arrow[dr, "p"'] & & E \arrow[dl, "q"] \\ & B & \end{tikzcd}$$
p, q are fibrations
f h.e.

Then f is h.e. over B.

$\exists\, g$ h.inverse to f. s.t.

$H': gf \sim id$ over B.

$H'': fg \sim id$ over B.

at any time t

$H'_t = E \xrightarrow{q} B$

$H''_t = D \xrightarrow{p} B$

prop.
$$\begin{tikzcd} D \arrow[r, "d"] \arrow[d, "p"'] & E \arrow[d, "q"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$
$p\ q$ fibration

$f\ d$ h.e.

then $(f\ d)$ a h.e. between pairs.

Based cofibrations & fibrations

everything is based.

$X \to Y$ based maps

- $[X, Y]$ = based maps $X \to Y / \sim$
- $\mathrm{Map}(X, Y)$ has a natural base pt

$$* : X \longrightarrow Y$$
$$x \longmapsto *$$

unbased:

$$\mathrm{Map}(X \times Y, Z) = \mathrm{Map}(X, \mathrm{Map}(Y, Z))$$

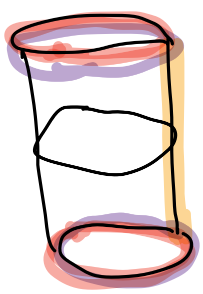
- based: map ⇝ based maps (sending base pt to base pt)

$$X \rightsquigarrow X \wedge Y = X \times Y / X \vee Y$$

$$X \vee Y = X \amalg Y / *_x \sim *_y$$

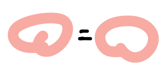
"coproduct in based spaces"

Ex: $S^1 \wedge S^1 = S^2$



$I / 0 \sim 1$

$I \times S^1 \overset{\text{identify}}{\rightsquigarrow} S^1 \times S^1 \overset{\text{quotient } S^1 \vee S^1}{\rightsquigarrow}$



In general $S^n \wedge S^m = S^{n+m}$

$\uparrow$ sphere of dim n

- adjunction in the based case

$$\mathrm{Map}(X \wedge Y, Z) = \mathrm{Map}(X, \mathrm{Map}(Y, Z))$$

(Map means based mapping spaces)

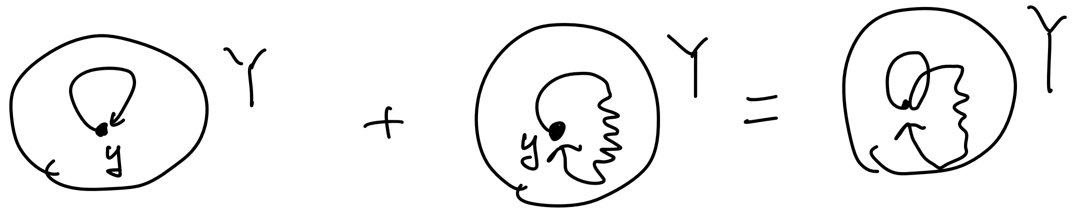
- notation: $\Sigma X := X \wedge S^1$
  $\Omega X := \mathrm{Map}(S^1, X)$

- $\pi_0(\mathrm{Map}(X, Y)) = [X, Y]$

prop: $[\Sigma X, Y]$ is a group;
$[\Sigma^2 X, Y]$ is an abelian group.

$$\begin{aligned} [\Sigma X, Y] &= \pi_0(\mathrm{Map}(\Sigma X, Y)) \\ &= \pi_0(\mathrm{Map}(X, \overbrace{\mathrm{Map}(S^1, Y)}^{\Omega Y})) \\ &= [X, \Omega Y] \end{aligned}$$

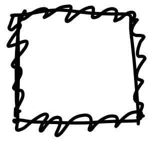
ΩY:



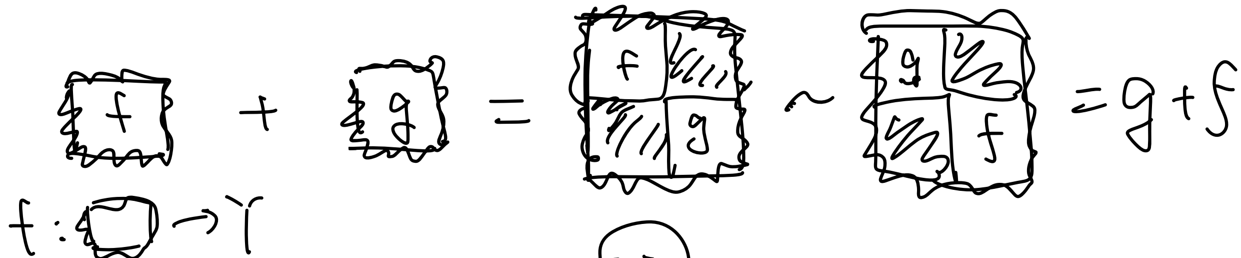
associative & unital up to homotopy

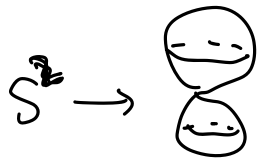
$\Omega^2 Y$: abelian up to homotopy

$\Omega^2 Y$: $S^2 \to Y$



$= S^2$





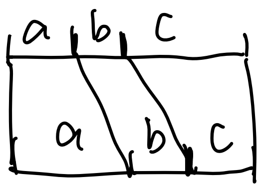
$\rightsquigarrow [X, \Omega^2 Y]$ abelian group str.

---

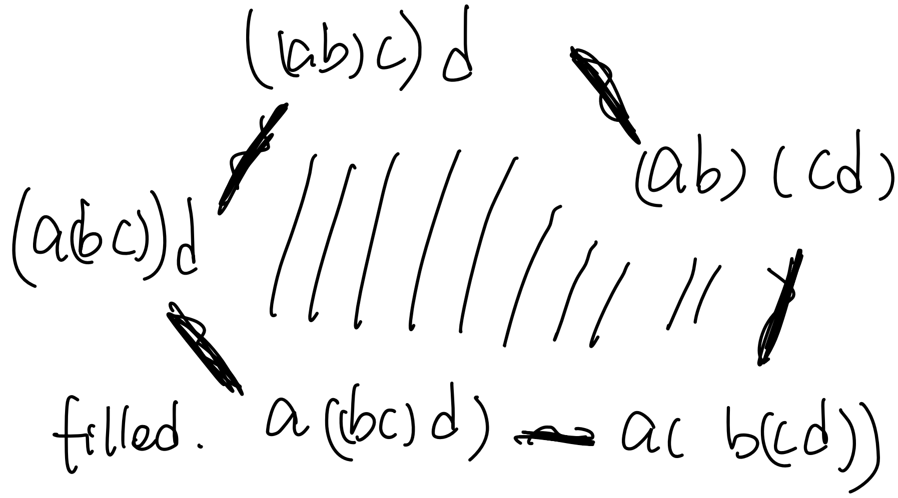
aside:

the addition on ΩY before passing to homotopy:

$(a+b)+c \sim a+(b+c)$



$a+b+c+d$



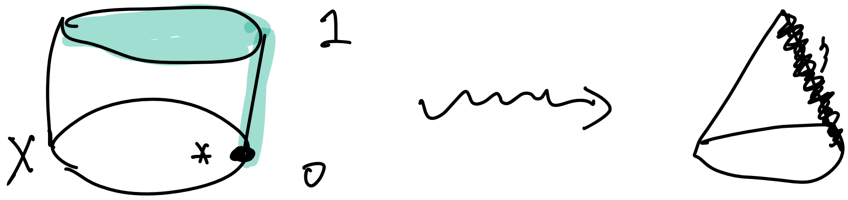
each vertex is a point in ΩY

& polygon can be filled.

add 5 things $\rightsquigarrow$ homotopy between htpy between htpy operad : str. to encode n-ary operations

---

- cone : $CX = X \times I / \{*\} \times I \cup X \times \{1\}$



- (based) cofibration (similar for based fibrations)

Def | replace everything in the unbased def with based spaces & maps.

$A \to X$ is a map based spaces.

$A \xrightarrow{i} X$ is a cofibration if

$\forall$

$$\begin{tikzcd} A \arrow[r, "i_0"] \arrow[d, hook, "i"'] & A \times I \arrow[dl, "H"'] \arrow[d, hook] \\ Y & \\ X \arrow[u, "f"] \arrow[r, "i_0"'] & X \times I \arrow[ul, dashed, "\exists \tilde{H}"'] \end{tikzcd}$$

, $\exists$ lift $\tilde{H}$

---

cat of based spaces

$X \in Top_*$

$uX \in Top$.

$Top_* \xrightarrow{u} Top$

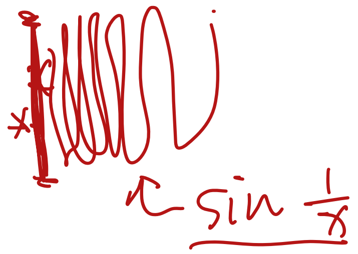
forget the base pt.

$A \to X$ based map.

If $uA \to uX$ is a cof $\rightsquigarrow$ $A \to X$ cof.

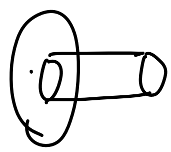
If $A \to X$ cof + A & X has non-degenerate base pt $\rightsquigarrow$ $uA \to uX$ is a cof

base pt: $* \to X$ is a cofibration



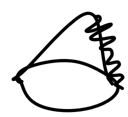
---

unbased

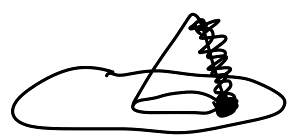
$Mf =$  $= Y \amalg_f X \times I$

based

$\rightsquigarrow$ • $Mf = Y \cup_f (X \wedge I_+)$



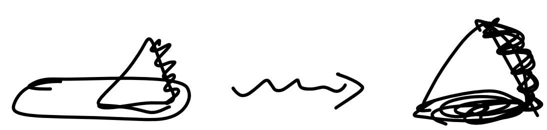
• $Cf = Y \cup_f CX$



prop: can replace any based map with cof + weak eq.

prop: $X \xrightarrow{f} Y \rightsquigarrow Cf/Y \overset{\text{homeo.}}{\cong} \Sigma X$

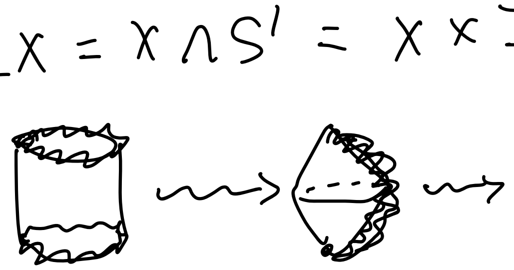
pf: exercise.

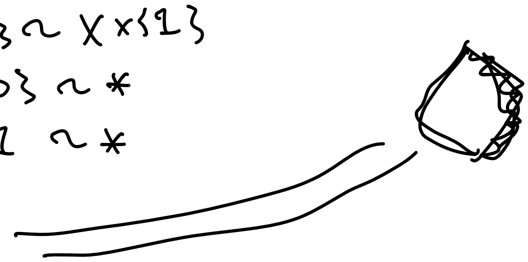


$\Sigma X = X \wedge S^1 = X \times I / X \times \{0\} \sim X \times \{1\}$

$X \times \{0\} \sim *$

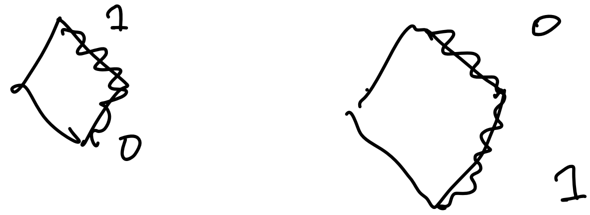
$* \times I \sim *$





⊛ $X \xrightarrow{f} Y \xrightarrow{i} C_f \xrightarrow{/Y} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \xrightarrow{-\Sigma i} \Sigma C_f \to \Sigma^2 X$
$\to \cdots$

$\Sigma X = X \wedge S^1 \ni (x,t)$ $\qquad (x,t) \longmapsto (f(x), 1-t)$



$$\begin{array}{ccc} X & \to & Y \\ & & \downarrow \\ & & Z \end{array}$$

Th'm $\forall$ based space Z, ⊛ induces a sequence

$$\underbrace{[X,Z] \xleftarrow{f^*} [Y,Z] \xleftarrow{i^*} [C_f,Z]}_{\text{sets}} \leftarrow \underbrace{[\Sigma X, Z] \leftarrow \cdots}_{\text{grps}}$$

~~grps~~ abelian grps

The sequence is exact.
ker = Im

Exactness at ☆ : (pf : next lecture)

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr] & Y \arrow[r, "i"] \arrow[d, "p"] & C_f \arrow[dl, dashed, "\exists"] \\ & Z & \end{tikzcd}$$

$p \in [Y,Z]$

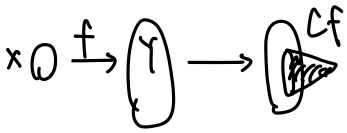
① p in the image i^* : $\exists$ ⇢ (i.e. $p \circ f \simeq *$)

② p in the ker f^* : $[p \circ f] = [*] \in [X,Z]$

Lemma 1

$f: X \to Y$ cof

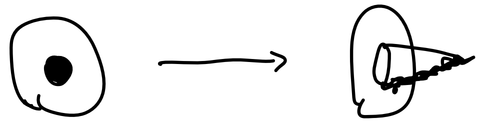
$Cf \xrightarrow{p} Cf/CX \cong Y/X$ p is a homotopy equivalence.



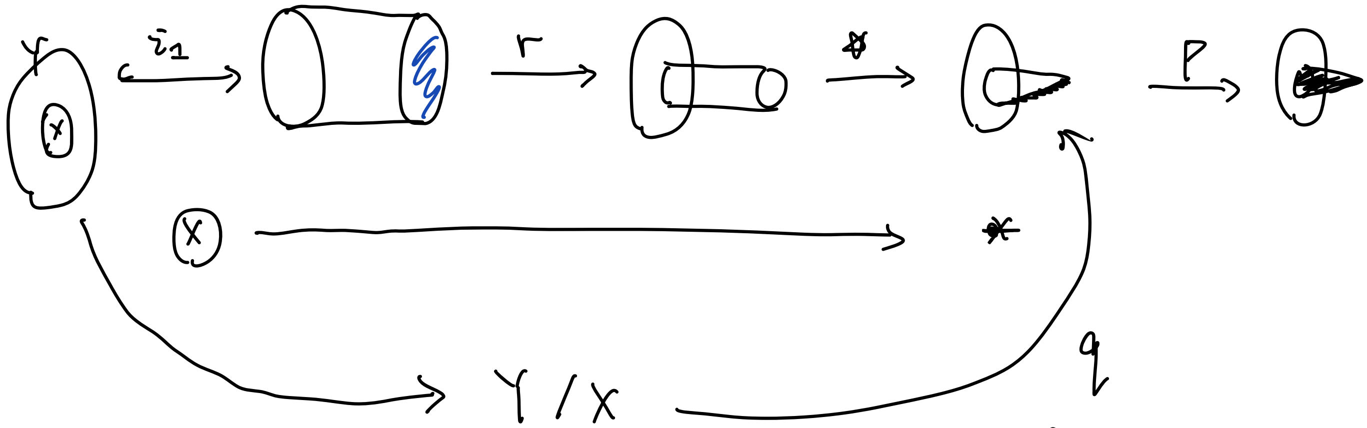
Pf sketch:

1) construct inverse

$q: Y/X \longrightarrow Cf$



f is cof $\Rightarrow \exists\, r$ s.t. $r|_{\ldots} = id$



$(\ast \circ r : X \times I \to \ldots)$

2) show $q \circ p \simeq id_{Cf}$: $+ (\ldots \times I \to \ldots)$

$p \circ q \simeq id_{Y/X}$: use r / … : … $\to$ …

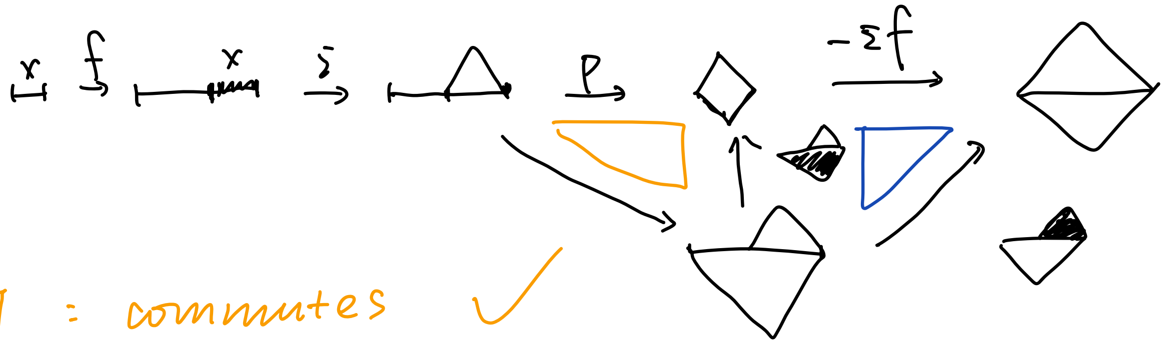
$Y/X \times I \to Y/X$

Lemma 2

$$X \xrightarrow{f} Y \xrightarrow{i} Cf \xrightarrow{p} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \longrightarrow$$

$$Ci$$

i is a cof



◁ : commutes ✓

▽ : commutes up to homotopy.

$[X,Z] \xleftarrow{f^*} [Y,Z] \xleftarrow{i^*} [Cf,Z] \leftarrow \cdots$ exact

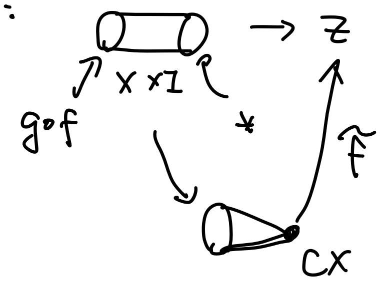
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g\circ f"'] & Y \arrow[r, "i"] \arrow[d, "g"] & Cf \arrow[dl, dashed, "\tilde g"] \\ & Z & \end{tikzcd}$$

$g\circ f \simeq * \iff g$ factors as $\tilde g \circ i$

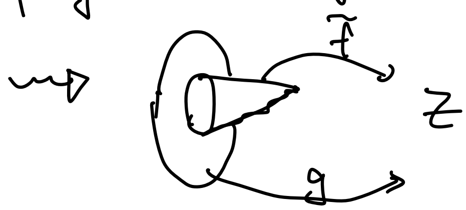
pf:

$\Rightarrow$: $X \xrightarrow{f} Y \xrightarrow{g} Z.$

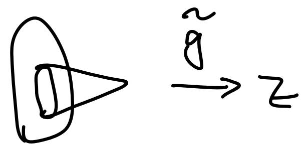
$H: g\circ f \to *$:



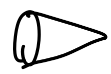
$\tilde f$ glued with g

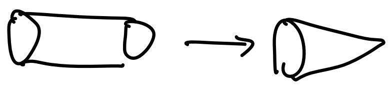


$\Leftarrow$

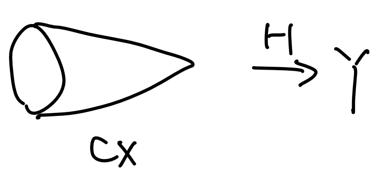


$\tilde g|_Y = g$

$\tilde g|_{CX} =$  $\to Z$

 $\xrightarrow{\tilde g|_{CX}} Z$: $gf \simeq *$

$f \overset{H}{\simeq} * : X \to Y$

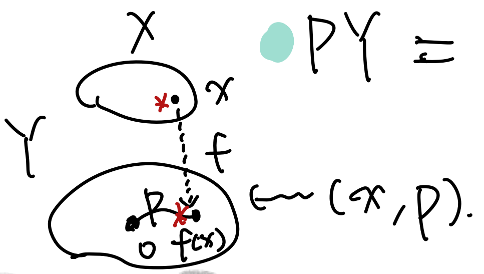


fiber sequences

$X \xrightarrow{f} Y$

- $Ff = X \times_f PY = \{ (x,p) \mid p(0) = * \in Y,\ f(x) = p(1) \}$

- $PY = \{ p \text{ path in } Y \mid p(0) = * \}$



- Ff: homotopy fiber
dual of Cf.

$$\begin{tikzcd} Ff \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\ PY \arrow[r] & Y \end{tikzcd} \qquad \begin{tikzcd} f^{-1}(*) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\ * \arrow[r] & Y \end{tikzcd}$$

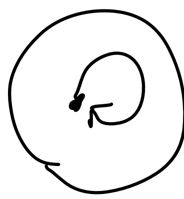
$Ff \simeq f^{-1}(*_y) \overset{\text{dual}}{\longleftrightarrow} Cf \simeq Y/X$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & CX \arrow[d] \\ Y \arrow[r] & Cf \arrow[ul, phantom, "\ulcorner"] \end{tikzcd} \rightarrow \begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ Y \arrow[r] & Y/X \end{tikzcd}$$

Have a sequence

$$(\circledast) \quad \cdots \to \Omega^2 Y \xrightarrow{-\Omega\iota} \Omega Ff \xrightarrow{-\Omega\pi_1} \Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{\iota} Ff \xrightarrow{\pi_1} X \xrightarrow{f} Y$$

$\Omega Y \cong F\pi_1$, $\quad Ff \cong f^{-1}(*)$



(x, p)

$p_0 = * \in Y$

$p_1 = f(x) \in Y$

$p \mapsto (*, p)$.

Thm $(\circledast)$ induces a sequence

$$\cdots \to [Z, \Omega Y] \to [Z, Ff] \to [Z, X] \to [Z, Y]$$

It is exact.

Adjoint :

$\eta : X \to \Omega \Sigma X.$ unit

$\varepsilon : \Sigma \Omega X \to X$ counit

$\eta : \quad x \mapsto (t \mapsto (x, t))$

$\varepsilon : (t, p) \mapsto p(t)$

$\eta : \quad Ff \to \Omega Cf$

$\varepsilon : \quad \Sigma Ff \to Cf.$

$$
\begin{tikzcd}
Ff \arrow[r, "\pi"] & X \arrow[r] \arrow[dr, bend right] & Y \arrow[r] & Cf \\
 & & C\pi \arrow[u, dashed] \arrow[r] & \Sigma Ff \arrow[u, dashed, "\varepsilon"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \Sigma\Omega Ff \arrow[r] \arrow[d] & \Sigma\Omega X \arrow[r] \arrow[d] & \Sigma\Omega Y \arrow[r] \arrow[d] & \Sigma Ff \arrow[r] \arrow[d] & \Sigma X \arrow[d, equal] \\
\Omega Y \arrow[r] \arrow[d, equal] & Ff \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Cf \arrow[r] \arrow[d] & \Sigma X \\
\Omega Y \arrow[r] & \Omega Cf \arrow[r] & \Omega\Sigma X \arrow[r] & \Omega\Sigma Y \arrow[r] & \Omega\Sigma Cf &
\end{tikzcd}
$$

Higher homotopy groups.

Def: n th homotopy group of X ← based.

$$\pi_n(X) := [S^n, X]$$

$n=1$: recovers the fundamental group

$n=0$: $\pi_0(X) = [S^0, X] = \#(\text{path connected components})$

Fact: $\pi_n(X) = \pi_{n-1}(\Omega X)$

pf: $[S^n, X] = \pi_0 \operatorname{Map}(S^n, X)$

$$\overset{S^n = S^1 \wedge S^{n-1}}{=} \pi_0 \operatorname{Map}(S^{n-1}, \underbrace{\operatorname{Map}(S^1, X)}_{= \Omega X})$$

$$= \pi_0 \operatorname{Map}(S^{n-1}, \Omega X)$$

$$= \pi_{n-1}(\Omega X)$$

Ex: $\pi_n X = \pi_0 \Omega^n X$

not always true $\quad \pi_n X \overset{\times}{=} \pi_{n+1} \Sigma X$

$E \xrightarrow{f} B$ a fibration $\quad F = f^{-1}(*_b)$

$$\cdots \to \Omega Ff \to \Omega E \to \Omega B \to \underset{\substack{\cong \\ F}}{Ff} \to E \to B$$

this gives a exact seq of homotopy grps

$Z = S^n$ + Th'm :

$$\begin{array}{c} [S^n, B] \\ \| \\ \pi_n B \end{array} \leftarrow \pi_n E \leftarrow \pi_n F \leftarrow \begin{array}{c} \pi_n \Omega B \\ \| \\ \pi_{n+1} B \end{array} \leftarrow \pi_{n+1} E \leftarrow \pi_{n+1} F$$

$$\leftarrow \pi_{n+2} B \leftarrow \cdots$$

<u>Ex</u>: 1) X is contractible.

$$\pi_n X = *$$

2) $E \to B$ a covering

$\leadsto$ $F \to E \to B$ a fiber sequence F is discrete

$\pi_n(F) = *$ $n \geq 1$

$\leadsto$ use LES : $\pi_0(F) = \pi_1(B) \leftarrow \pi_1(E) \leftarrow \pi_1(F) \leftarrow \pi_2(B)$

$$\pi_0(F) = \#(F), \quad \pi_1(F) = 0, \quad \pi_2(B) \leftarrow \pi_2(E) \leftarrow \pi_2(F) = 0 \leftarrow$$

$\Longrightarrow$ $\pi_n(B) = \pi_n(E)$ $n \geq 2$

3) $$\pi_n(S^1) = \begin{cases} \mathbb{Z} & n = 1 \\ 0 & n \geq 2 \end{cases}$$

use $\mathbb{R} \to S^1$

$$\pi_n(\mathbb{R}) = \pi_n(S^1) \quad n \geq 2$$

4) $\pi_n(S^m) = 0$ for $n < m$

1) ~~trasenvesality~~ transversality $\rightsquigarrow$ $S^n \xrightarrow{f} S^m$ missing a point

i.e.
$$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[dr, "\tilde{f}"'] & S^m \\ & S^m \setminus \{p\} \arrow[u, leftrightarrow] \end{tikzcd}$$
$$S^m \setminus \{p\} = \mathbb{R}^m$$

$$* \overset{H}{\sim} \tilde{f} : S^n \to \mathbb{R}^m \Rightarrow f \overset{i \circ H}{\sim} *$$

5) $S^1 \to S^3 \to S^2$

$$\underset{=0}{\pi_1(S^2)} \leftarrow \underset{=0}{\pi_1(S^3)} \leftarrow \underset{=\mathbb{Z}}{\pi_1(S^1)} \leftarrow \pi_2(S^2) \leftarrow \underset{=0}{\pi_2(S^3)}$$

$$\pi_4(S^2) \to \underset{=0}{\pi_3(S^1)} \to \pi_3(S^3) \to \pi_3(S^2) \to \underset{=0}{\pi_2(S^1)}$$

$\Rightarrow$ $\pi_2(S^2) = \mathbb{Z}$

$\pi_n(S^2) = \pi_n(S^3)$ $n \geq 3$