Nov 13

# Universal coefficient th'm
# Künneth formular

## 1) Universal coefficient th'm

Q: $H_*(X; R)$ known.

how to compute $H_*(X; M)$ ?

$$
\begin{array}{lcccc}
 & * = & 0 & 1 & 2 \\
\text{Ex. (last time)} \quad H_*(\mathbb{RP}^2; \mathbb{Z}) & = & \mathbb{Z}, & \mathbb{Z}/2, & 0 \\
H_*(\mathbb{RP}^2; \mathbb{Z}/2) & = & \mathbb{Z}/2, & \mathbb{Z}/2, & \mathbb{Z}/2
\end{array}
$$

$\longrightarrow H_*(X; \mathbb{Z}/2) \neq H_*(X; \mathbb{Z}) \otimes \mathbb{Z}/2$

$$C_*(X) \otimes \mathbb{Z}/2 = C_*(X; \mathbb{Z}/2)$$

Compare : $H_*(C_*(X)) \otimes M$ with $H_*(C_*(X) \otimes M)$

In $\mathbb{RP}^2$ case:

$$C_*(\mathbb{RP}^2) = \overset{0}{\mathbb{Z}} \xleftarrow{\times 0} \overset{1}{\mathbb{Z}} \xleftarrow{\times 2} \overset{2}{\mathbb{Z}}$$

$$C_*(\mathbb{RP}^2) \otimes \mathbb{Z}/2 = \mathbb{Z}/2 \xleftarrow{\times 0} \mathbb{Z}/2 \xleftarrow{\times 0} \mathbb{Z}/2$$

$- \otimes M$ is not exact.

$$0 \to A \to B \to C \to 0$$

$$\cancel{0 \to} A \otimes M \to B \otimes M \to C \otimes M \to 0.$$

$\mathrm{Tor}_*^{\mathbb{Z}}(-, M)$ measure the failure of $- \otimes M$ being exact

$$\cdots \to \mathrm{Tor}_1^{\mathbb{Z}}(C, M) \to A \otimes M \to B \otimes M \to C \otimes M \to 0$$

How to compute $\mathrm{Tor}_*^{\mathbb{Z}}(C, M)$

1) find a free resolution $F_*$ of $M$

$M = \mathbb{Z}/p \rightsquigarrow F_* = \mathbb{Z} \xrightarrow{\times p} \mathbb{Z}$ $\quad$ ($\mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \to M$ is exact)

$\to$ a chain cplx $C_*$

each level is free $\mathbb{Z}$-module

$H_*(C_*) = M \quad * = 0.$

2) $C \otimes F_*$

3) $H_*(C \otimes F_*) = \mathrm{Tor}_*^{\mathbb{Z}}(C, M)$

$\mathrm{Tor}_*$ : derived functor of $\otimes$

$C \otimes M$ derived functor : 1) replace $M$ with free resolution

2) $C \otimes$ replacement

● properties of Tor

1) $\mathrm{Tor}_*^{\mathbb{Z}}(C, M) = 0$ for $* \geq 2$

Reason : $M$ can be resolved in 2 steps.

eg. $M = \mathbb{Z}/p$. ⇝ $\overset{1}{\mathbb{Z}} \xrightarrow{\times p} \overset{0}{\mathbb{Z}}$

⇝ $\overset{1}{\mathbb{Z} \otimes C} \xrightarrow{\times p} \overset{0}{\mathbb{Z} \otimes C}$

⇝ $H_*(\overset{1}{\mathbb{Z} \otimes C} \xrightarrow{\times p} \overset{0}{\mathbb{Z} \otimes C}) = \mathrm{Tor}_*^{\mathbb{Z}}(C, M)$

$C = \mathbb{Z}$ : $H_*(\mathbb{Z} \xrightarrow{\times p} \mathbb{Z}) = \overset{1}{0} \quad \overset{0}{\mathbb{Z}/p}$ $\quad \mathrm{Tor}_*^{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/p) = \begin{cases} \mathbb{Z}/p & 0 \\ 0 & \text{o.w.} \end{cases}$

$C = \mathbb{Z}/p$ : $H_*(\mathbb{Z} \otimes \mathbb{Z}/p \xrightarrow{\times p} \mathbb{Z} \otimes \mathbb{Z}/p)$

$= H_*(\mathbb{Z}/p \xrightarrow[\times 0]{\times p \, \|} \mathbb{Z}/p) = \overset{1}{\mathbb{Z}/p} \quad \overset{0}{\mathbb{Z}/p}$

$\mathrm{Tor}_*^{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}/p) = \begin{cases} \mathbb{Z}/p & 1 \\ \mathbb{Z}/p & 0 \\ 0 & \text{o.w} \end{cases}$

prop :

1). $\mathrm{Tor}_*^{\mathbb{Z}}(C, \mathbb{Z}/p) = \begin{cases} C/pC & 0 \\ p\text{-torsions.} & 1 \\ 0 & * \geq 2 \end{cases}$

$H_*(C \xrightarrow{\times p} C) = \begin{cases} C/pC & 0 \\ p\text{-tor.} & 1 \end{cases}$

2) $\mathrm{Tor}_*^{\mathbb{Z}}(C, \mathbb{Z}) = C \otimes \mathbb{Z} \quad 0$

$- \otimes \mathbb{Z}$ is exact

Th'm, (UCT) $X_\bullet$ is a flat chain complex.

$$0 \to H_n(X) \otimes M \to H_n(X_\bullet; M) \to \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(X), M) \to 0.$$

( $X_\bullet$ is a cellular chain for some space ⇝ $X$ is free chain cplx ⇝ $X$ is flat )

Pf : $H_*(X_\bullet \otimes M) \overset{?}{\longleftrightarrow} H_*(X_\bullet) \otimes M$

If $M$ free $\mathbb{Z}$-module : $? = \cong$

If $M$ not free : resolution of $M$: $F_1 \to F_0$.

$( 0 \to \underset{\text{free}}{\underset{\uparrow}{F_1}} \to \underset{\text{free.}}{\underset{\uparrow}{F_0}} \to M \to 0$ exact )

$0 \to F_1 \to F_0 \to M \to 0$ SES

$$\begin{tikzcd}[column sep=small]
 & \cdots \arrow[r] & H_n(X_\bullet \otimes F_1) \arrow[r] & \overset{\text{coker}}{H_n(X_\bullet \otimes F_0)} \arrow[r] & H_n(X_\bullet \otimes M) \arrow[dll, rounded corners] \\
 & & \underset{\text{ker}}{H_{n-1}(X_\bullet \otimes F_1)} \arrow[r] & H_{n-1}(X_\bullet \otimes F_0) \arrow[r] & \cdots
\end{tikzcd}$$

$\rightsquigarrow$ (LES)

$$\begin{tikzcd}[column sep=small]
\arrow[r] & A \arrow[r, "f"] & B \arrow[r] & C \arrow[r] \arrow[d, squiggly] & D \arrow[r, "g"] & E \arrow[r] & {}. \\
0 \arrow[r] & \operatorname{coker} f \arrow[rr] & & C \arrow[rr] & & \ker g \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
0 \arrow[d] & H_*(X_\bullet) \otimes M \arrow[dr, equal] & 0 \arrow[d] \\
\operatorname{coker}(H_*(X_\bullet \otimes F_1) \to H_*(X_\bullet \otimes F_0)) \arrow[d] \arrow[rr, phantom, "\cong"] & & \operatorname{coker}(H_*(X_\bullet) \otimes F_1 \to H_*(X_\bullet) \otimes F_0) \arrow[d] \\
H_*(X_\bullet \otimes M) \arrow[d] \arrow[rr, equal] & & H_*(X_\bullet \otimes M) \arrow[d] \\
\ker(H_{*-1}(X_\bullet \otimes F_1) \to H_{*-1}(X_\bullet \otimes F_0)) \arrow[d] \arrow[rr, phantom, "\cong"] & & \ker(H_{*-1}(X_\bullet) \otimes F_1 \to H_{*-1}(X_\bullet) \otimes F_0) \arrow[d] \\
0 & \mathrm{Tor}_1^{\mathbb{Z}}(H_{*-1}(X_\bullet), M) \arrow[ur, equal] \arrow[d, equal] & 0 \\
 & H_1(H_{*-1}(X_\bullet) \otimes F_1 \to H_{*-1}(X_\bullet) \otimes F_0) &
\end{tikzcd}$$

E.g. $X = \mathbb{R}P^2$ $\quad H_*(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & 0 \\ \mathbb{Z}/2 & 1 \\ 0 & 2 \end{cases}$

$$0 \to H_*(X) \otimes \mathbb{Z}/2 \longrightarrow H_*(X;\mathbb{Z}/2) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(H_{*-1}(X), \mathbb{Z}/2) \to 0.$$

$* = 0.$ $\quad 0 \to \mathbb{Z}/2 \to ?? \to 0 \to 0 \rightsquigarrow H_0(X;\mathbb{Z}/2) = \mathbb{Z}/2.$

$* = 1$ $\quad 0 \to \mathbb{Z}/2 \to ?? \to \underset{\rightsquigarrow 0}{\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/2)} \to 0 \rightsquigarrow H_1(X;\mathbb{Z}/2) = \mathbb{Z}/2$

$* = 2$ $\quad 0 \to 0 \to ?? \to \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}/2) \overset{= \mathbb{Z}/2}{} \to 0 \rightsquigarrow H_2(X;\mathbb{Z}/2) = \mathbb{Z}/2.$

2) Künneth formula.

UCT: $H_*(X)\otimes M \overset{??}{\longleftrightarrow} H_*(X\otimes M)$.

$M$: $\mathbb{Z}$-module / chain cplx concentrated in deg=0

$\leftarrow 0 \leftarrow 0 \leftarrow \underset{0}{M} \leftarrow 0 \leftarrow 0 \leftarrow$

Q: what if $M$ is a general chain cplx?

$C_\bullet$ $D_\bullet$ chain cplx.

$$H_*(C_\bullet\otimes D_\bullet) \overset{??}{\longleftrightarrow} H_*(C_\bullet)\otimes H_*(D_\bullet)$$

① Th'm.
$$0\to \bigoplus_{p+q=n} H_p(C)\otimes H_q(D) \to H_n(C_\bullet\otimes D_\bullet) \to \bigoplus_{p+q=n-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_p(C), H_q(D)) \to 0.$$

② Th'm $H_n(X\times Y) \simeq H_n(C_\bullet(X)\otimes C_\bullet(Y))$

①+② $\Rightarrow$ Künneth formula

## Nov 15

comments on UCT

1) work over $k$: $\mathrm{Tor}^k$ vanishes

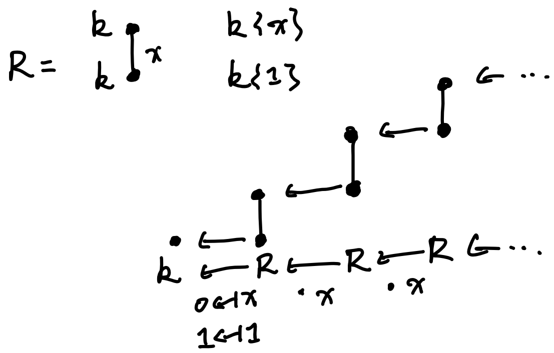
$$0\to H_*(X;\mathbb{Z})\otimes M \to H_*(X;M) \to \mathrm{Tor}_1^{\mathbb{Z}}(H_{*-1}(X), M)\to 0.$$

replace $\mathbb{Z}$ with some ring $R$ ← principle ideal domain ← can be resolved in 2 steps.

2) for $R=$ PID $\mathrm{Tor}_*^R = 0$ for $* \geq 2$

ex. $R = k[x]/x^2$ ($k$: field)

resolve $k$



$\cdot x$ : $x \mapsto 0$

$1 \mapsto x$

can't resolve in finitely many steps.

3) $\mathrm{Tor}^R_*(A, B)$.

1) free resolution $M_\bullet$ for $A$

2) $H_*(M_\bullet \otimes B)$.

$\mathrm{Tor}^R_*(A, B)$ independent of the choice $M_\bullet$

Th'm (fundamental th'm in homological alg.)

$A \leftarrow M_0 \leftarrow M_1 \leftarrow \cdots$ free resolution

$A' \leftarrow N_0 \leftarrow N_1 \leftarrow \cdots$ another resolution

$f: A \to A'$ a module map.

Then $f$ lifts to $M_\bullet \to N_\bullet$, unique up to chain homotopy.

4).

$$0 \to H_*(X;\mathbb{Z}) \otimes M \to H_*(X;M) \to \mathrm{Tor}_1^{\mathbb{Z}}(H_{*-1}(X), M) \to 0.$$

This splits, but not naturally.

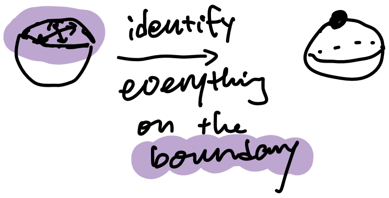
split: $0 \to A \to B \to C \to 0$

$B = A \oplus C$ ( non ex. $0 \to \mathbb{Z}/2 \xrightarrow{\times 2} \mathbb{Z}/4 \xrightarrow{q} \mathbb{Z}/2 \to 0$

$\mathbb{Z}/4 \neq \mathbb{Z}/2 \oplus \mathbb{Z}/2$ )

not naturally:

$\mathbb{R}P^2 \to S^2$



$$\begin{tikzcd}
 & 0 \arrow[d, equal] & \mathbb{Z}/2 \arrow[d, equal] & \mathbb{Z}/2 \arrow[d, equal] & \\
0 \arrow[r] & H_2(\mathbb{R}P^2) \otimes \mathbb{Z}/2 \arrow[r] \arrow[d, "0"'] & H_2(\mathbb{R}P^2; \mathbb{Z}/2) \arrow[r] \arrow[d, "\cong"] & \mathrm{Tor}_1^{\mathbb{Z}}(H_1(\mathbb{R}P^2), \mathbb{Z}/2) \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & H_2(S^2) \otimes \mathbb{Z}/2 \arrow[r] \arrow[d, equal] & H_2(S^2; \mathbb{Z}/2) \arrow[r] \arrow[d, equal] & \mathrm{Tor}_1^{\mathbb{Z}}(H_1(S^2), \mathbb{Z}/2) \arrow[r] \arrow[d, equal] & 0 \\
 & \mathbb{Z}/2 & \mathbb{Z}/2 & 0 &
\end{tikzcd}$$

2). Künneth formula.

$$H_*(X\times Y) \overset{??}{\longleftrightarrow} H_*(X)\otimes H_*(Y)$$

$$\begin{tikzcd}
H_*(C_*(X\times Y)) \arrow[dr, hook] & & H_*(C_*(X))\otimes H_*(C_*(Y)) \arrow[dl, leftrightarrow] \\
& H_*(C_*(X)\otimes C_*(Y)) &
\end{tikzcd}$$

generalized version of $\underline{UCT}$ $\uparrow$ special case. $C_*(Y) = \overset{\overset{0}{\shortparallel}}{\underset{\downarrow}{d}}\ M$

( Some basics about chain cplx).

$$C_\bullet = \cdots \leftarrow C_0 \xleftarrow{d_1} C_1 \xleftarrow{d_2} C_2 \leftarrow \cdots \qquad d^2 = 0$$

$$(A_\bullet \otimes B_\bullet)_n = \bigoplus_{p+q=n} A_p \otimes B_q$$

$$d(a\otimes b) = d(a)\otimes b + (-1)^{\deg(a)}\, a\otimes d(b)$$

$$d^2(a\otimes b) = d(d(a)\otimes b + (-1)^{\deg(a)}\, a\otimes d(b))$$

$$\boxed{\begin{matrix} a\in A_p \\ \deg(a) = p\end{matrix}} \qquad = d^2\cancel{(a)}\otimes b + (-1)^{\deg a-1} d(a)\otimes d(b)$$

$$+ (-1)^{\deg(a)} d(a)\otimes d(b) + (-1)^{\deg(a)+\deg(a)}\, \cancel{a\otimes} d^2(b) \qquad = 0$$

cancel

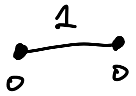
Chain homotopy.

$$A_\bullet \underset{g}{\overset{f}{\rightrightarrows}} B_\bullet \quad \text{map of chain cplx} : \begin{matrix}\exists f_n : A_n \to B_n \} \text{ s.t.} \\ f_n d = d f_n .\end{matrix}$$

f is chain homotopic to g if.

$$H: A_\bullet \to B_{\bullet+1} \quad \text{s.t.} \quad dH + Hd = f-g$$

$$I_\bullet = 0 \to 0 \to 0 \to \underset{1}{\mathbb{Z}} \xrightarrow{(1,-1)} \underset{0}{\mathbb{Z}\oplus\mathbb{Z}}$$



Claim: H same information as.

$$\begin{tikzcd}
A_\bullet \arrow[d] \arrow[dr, bend left, "f"] & \\
A_\bullet\otimes I_\bullet \arrow[r, "(f,g,H)"] & B_\bullet \\
A_\bullet \arrow[u] \arrow[ur, "g"'] &
\end{tikzcd}$$

commutative diagram.

$$(A_\bullet\otimes I_\bullet)_n = A_n \oplus A_n \oplus A_{n-1} \xrightarrow{(f,g,H)} B_n$$

$$\bigoplus_{p+q=n} A_p\otimes I_q$$

check: commutativity ⇔ $\begin{matrix} dH+Hd \\ \| \\ f-g \end{matrix}$

Def: $A_\bullet \xrightarrow{f} B_\bullet$ f is a quasi isomorphism if $H_*(f)$ is an isomorphism.

Def: $A_\bullet \underset{g}{\overset{f}{\rightleftarrows}} B_\bullet$ f is an isomorphism if

∃ g s.t. $f\circ g$ chain homotopic to $id_{B_\bullet}$

$g\circ f$ ˅ $id_{A_\bullet}$

| space | chain cplx |
|---|---|
| homotopy | chain homotopy |
| homotopy equivalence | chain isomorphism |
| weak equivalence | quasi isomorphism |
| $\pi_\bullet$ | $H_\bullet$ |

## Künneth formula.

$$\begin{tikzcd}
H_*(C_*(X\times Y)) \arrow[dr, squiggly, "\textcircled{2}"'] & & H_*(C_*(X))\otimes H_*(C_*(Y)) \\
& H_*(C_*(X)\otimes C_*(Y)) \arrow[ur, "\textcircled{1}"'] &
\end{tikzcd}$$

① : similar to UCT.

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{p+q=n} H_p(C_*(X)) \otimes H_q(C_*(Y)) \arrow[r] & H_n(C_*(X) \otimes C_*(Y)) \arrow[r] & \bigoplus_{p+q=n-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_p(C_*(X)), H_q(C_*(Y))) \arrow[r] & 0 \\
& & H_n(C_*(X \times Y)) \arrow[u, "\wr", "②"'] & &
\end{tikzcd}$$

② cellular chain :

$$C_*(X) \otimes C_*(Y) \xrightarrow{K} C_*(X \times Y)$$

$$[i] \otimes [j] \longmapsto (-1)^{pq} [i \times j]$$

$i$ : $p$-cell
$j$ : $q$-cell

$K$ : isomorphism of chain cplx.

remarks :

1). over $k$ : no Tor.

(künneth isomorphism)

$$H_*(X \times Y ; k) \cong H_*(X ; k) \otimes H_*(Y ; k).$$

2) split, but not natural.

3) ② : singular chain :

Eilenberg-Zilber th'm.

$$\begin{tikzcd}
S_*(X \times Y) \arrow[r, shift left, "F"] & S_*(X) \otimes S_*(Y) \arrow[l, shift left, "G"]
\end{tikzcd}$$

Eilenberg-Zilber th'm : $\exists$ $F$, $G$.

s.t. $FG = id$

$GF \underset{\text{chain homotopic}}{\simeq} id$.

Ex. $H_*(\mathbb{RP}^3 \times \mathbb{RP}^3)$ with $\mathbb{Z}$ coeff.

$\mathbb{Z}/2$ coeff.

1) $\mathbb{Z}/2$ coefficient : field $\Rightarrow$ Künneth iso.

$$H_*(\mathbb{RP}^3 \times \mathbb{RP}^3 ; \mathbb{Z}/2) = H_*(\mathbb{RP}^3 ; \mathbb{Z}/2) \otimes H_*(\mathbb{RP}^3 ; \mathbb{Z}/2)$$

chain $\mathbb{RP}^3$

$$\begin{array}{cccccccc} 0 & & 1 & & 2 & & 3 & \\ \mathbb{Z} & \xleftarrow[0]{} & \mathbb{Z} & \xleftarrow[2]{} & \mathbb{Z} & \xleftarrow[0]{} & \mathbb{Z} & 0 \cdots\cdots \end{array}$$

$H_*(\mathbb{RP}^3 ; \mathbb{Z}/2) = \mathbb{Z}/2 \quad \mathbb{Z}/2 \quad \mathbb{Z}/2 \quad \mathbb{Z}/2$

2) $H_*(\mathbb{RP}^3) = \begin{array}{cc} \mathbb{Z} & 3 \\ 0 & 2 \\ \mathbb{Z}/2 & 1 \\ \mathbb{Z} & 0 \end{array}$

$\mathrm{Tor}_1^{\mathbb{Z}}(H_1(\mathbb{RP}^3), H_1(\mathbb{RP}^3)) = \mathbb{Z}/2$ only non trivial term

$\Downarrow$

contribute to $H_3(\mathbb{RP}^3 \times \mathbb{RP}^3 ; \mathbb{Z})$.

$$H_n(\mathbb{RP}^3 \times \mathbb{RP}^3) = \begin{cases} \bigoplus_{p+q=n} H_p(\mathbb{RP}^3) \otimes H_q(\mathbb{RP}^3) & n \neq 3. \\ \bigoplus_{p+q=3} H_p(\mathbb{RP}^3) \otimes H_q(\mathbb{RP}^3) \oplus \mathbb{Z}/2 & n = 3 \end{cases}$$