Chapter 2

Homology

The fundamental group $\pi_1(X)$ is especially useful when studying spaces of low dimension, as one would expect from its definition which involves only maps from low-dimensional spaces into X, namely loops $I \to X$ and homotopies of loops, maps $I \times I \to X$. The definition in terms of objects that are at most 2-dimensional manifests itself for example in the fact that when X is a CW complex, $\pi_1(X)$ depends only on the 2-skeleton of X. In view of the low-dimensional nature of the fundamental group, we should not expect it to be a very refined tool for dealing with high-dimensional spaces. Thus it cannot distinguish between spheres S^n with $n \geq 2$. This limitation to low dimensions can be removed by considering the natural higher-dimensional analogs of $\pi_1(X)$, the homotopy groups $\pi_n(X)$, which are defined in terms of maps of the n-dimensional cube I^n into X and homotopies $I^n \times I \to X$ of such maps. Not surprisingly, when X is a CW complex, $\pi_n(X)$ depends only on the $(n+1)$-skeleton of X. And as one might hope, homotopy groups do indeed distinguish spheres of all dimensions since $\pi_i(S^n)$ is 0 for $i < n$ and $\mathbb{Z}$ for $i = n$.

However, the higher-dimensional homotopy groups have the serious drawback that they are extremely difficult to compute in general. Even for simple spaces like spheres, the calculation of $\pi_i(S^n)$ for $i > n$ turns out to be a huge problem. Fortunately there is a more computable alternative to homotopy groups: the homology groups $H_n(X)$. Like $\pi_n(X)$, the homology group $H_n(X)$ for a CW complex X depends only on the $(n+1)$-skeleton. For spheres, the homology groups $H_i(S^n)$ are isomorphic to the homotopy groups $\pi_i(S^n)$ in the range $1 \leq i \leq n$, but homology groups have the advantage that $H_i(S^n) = 0$ for $i > n$.

The computability of homology groups does not come for free, unfortunately. The definition of homology groups is decidedly less transparent than the definition of homotopy groups, and once one gets beyond the definition there is a certain amount of technical machinery to be set up before any real calculations and applications can be given. In the exposition below we approach the definition of $H_n(X)$ by two preliminary stages, first giving a few motivating examples nonrigorously, then constructing a restricted model of homology theory called simplicial homology, before plunging into the general theory, known as singular homology. After the definition of singular homology has been assimilated, the real work of establishing its basic properties begins. This takes close to 20 pages, and there is no getting around the fact that it is a substantial effort. This takes up most of the first section of the chapter, with small digressions only for two applications to classical theorems of Brouwer: the fixed point theorem and 'invariance of dimension'.

The second section of the chapter gives more applications, including the homology definition of Euler characteristic and Brouwer's notion of degree for maps $S^n \to S^n$. However, the main thrust of this section is toward developing techniques for calculating homology groups efficiently. The maximally efficient method is known as cellular homology, whose power comes perhaps from the fact that it is 'homology squared' — homology defined in terms of homology. Another quite useful tool is Mayer–Vietoris sequences, the analog for homology of van Kampen's theorem for the fundamental group.

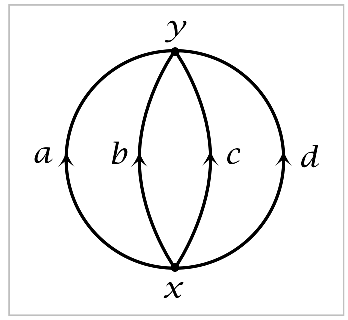
An interesting feature of homology that begins to emerge after one has worked with it for a while is that it is the basic properties of homology that are used most often, and not the actual definition itself. This suggests that an axiomatic approach to homology might be possible. This is indeed the case, and in the third section of the chapter we list axioms which completely characterize homology groups for CW complexes. One could take the viewpoint that these rather algebraic axioms are all that really matters about homology groups, that the geometry involved in the definition of homology is secondary, needed only to show that the axiomatic theory is not vacuous. The extent to which one adopts this viewpoint is a matter of taste, and the route taken here of postponing the axioms until the theory is well-established is just one of several possible approaches.

The chapter then concludes with three optional sections of Additional Topics. The first is rather brief, relating $H_1(X)$ to $\pi_1(X)$, while the other two contain a selection of classical applications of homology. These include the n-dimensional version of the Jordan curve theorem and the 'invariance of domain' theorem, both due to Brouwer, along with the Lefschetz fixed point theorem.

The Idea of Homology

The difficulty with the higher homotopy groups π_n is that they are not directly computable from a cell structure as π_1 is. For example, the 2-sphere has no cells in dimensions greater than 2, yet its n-dimensional homotopy group $\pi_n(S^2)$ is nonzero for infinitely many values of n. Homology groups, by contrast, are quite directly related to cell structures, and may indeed be regarded as simply an algebraization of the first layer of geometry in cell structures: how cells of dimension n attach to cells of dimension $n-1$.

Let us look at some examples to see what the idea is. Consider the graph X_1 shown in the figure, consisting of two vertices joined by four edges. When studying the fundamental group of X_1 we consider loops formed by sequences of edges, starting and ending at a fixed basepoint. For example, at the basepoint x, the loop ab^{-1} travels forward along the edge a, then backward along b, as indicated by the exponent -1. A more complicated loop would be $ac^{-1}bd^{-1}ca^{-1}$. A salient feature of the fundamental group is that it is generally nonabelian, which both enriches and complicates the theory. Suppose we simplify matters by abelianizing. Thus for example the two loops ab^{-1} and $b^{-1}a$ are to be regarded as equal if we make a commute with b^{-1}. These two loops ab^{-1} and $b^{-1}a$ are really the same circle, just with a different choice of starting and ending point: x for ab^{-1} and y for $b^{-1}a$. The same thing happens for all loops: Rechoosing the basepoint in a loop just permutes its letters cyclically, so a byproduct of abelianizing is that we no longer have to pin all our loops down to a fixed basepoint. Thus loops become *cycles*, without a chosen basepoint.



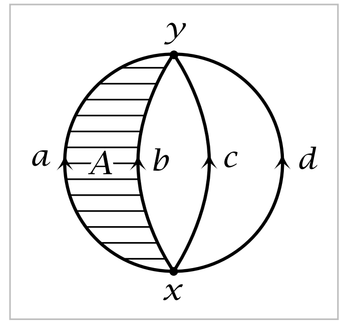
Having abelianized, let us switch to additive notation, so cycles become linear combinations of edges with integer coefficients, such as $a - b + c - d$. Let us call these linear combinations *chains* of edges. Some chains can be decomposed into cycles in several different ways, for example $(a - c) + (b - d) = (a - d) + (b - c)$, and if we adopt an algebraic viewpoint then we do not want to distinguish between these different decompositions. Thus we broaden the meaning of the term ‘cycle’ to be simply any linear combination of edges for which at least one decomposition into cycles in the previous more geometric sense exists.

What is the condition for a chain to be a cycle in this more algebraic sense? A geometric cycle, thought of as a path traversed in time, is distinguished by the property that it enters each vertex the same number of times that it leaves the vertex. For an arbitrary chain $ka + \ell b + mc + nd$, the net number of times this chain enters y is $k + \ell + m + n$ since each of a, b, c, and d enters y once. Similarly, each of the four edges leaves x once, so the net number of times the chain $ka + \ell b + mc + nd$ enters x is $-k - \ell - m - n$. Thus the condition for $ka + \ell b + mc + nd$ to be a cycle is simply $k + \ell + m + n = 0$.

To describe this result in a way that would generalize to all graphs, let C_1 be the free abelian group with basis the edges a, b, c, d and let C_0 be the free abelian group with basis the vertices x, y. Elements of C_1 are chains of edges, or 1-dimensional chains, and elements of C_0 are linear combinations of vertices, or 0-dimensional chains. Define a homomorphism $\partial : C_1 \to C_0$ by sending each basis element a, b, c, d to $y - x$, the vertex at the head of the edge minus the vertex at the tail. Thus we have $\partial(ka + \ell b + mc + nd) = (k + \ell + m + n)y - (k + \ell + m + n)x$, and the cycles are precisely the kernel of ∂. It is a simple calculation to verify that $a - b$, $b - c$, and $c - d$

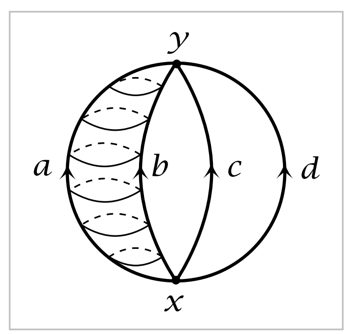
form a basis for this kernel. Thus every cycle in X_1 is a unique linear combination of these three most obvious cycles. By means of these three basic cycles we convey the geometric information that the graph X_1 has three visible 'holes', the empty spaces between the four edges.

Let us now enlarge the preceding graph X_1 by attaching a 2-cell A along the cycle $a - b$, producing a 2-dimensional cell complex X_2. If we think of the 2-cell A as being oriented clockwise, then we can regard its boundary as the cycle $a - b$. This cycle is now homotopically trivial since we can contract it to a point by sliding over A. In other words, it no longer encloses a hole in X_2. This suggests that we form a quotient of the group of cycles in the preceding example by factoring out the subgroup generated by $a - b$. In this quotient the cycles $a - c$ and $b - c$, for example, become equivalent, consistent with the fact that they are homotopic in X_2.



Algebraically, we can define now a pair of homomorphisms $C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$ where C_2 is the infinite cyclic group generated by A and $\partial_2(A) = a - b$. The map ∂_1 is the boundary homomorphism in the previous example. The quotient group we are interested in is $\operatorname{Ker}\partial_1 / \operatorname{Im}\partial_2$, the kernel of ∂_1 modulo the image of ∂_2, or in other words, the 1-dimensional cycles modulo those that are boundaries, the multiples of $a - b$. This quotient group is the *homology group* $H_1(X_2)$. The previous example can be fit into this scheme too by taking C_2 to be zero since there are no 2-cells in X_1, so in this case $H_1(X_1) = \operatorname{Ker}\partial_1 / \operatorname{Im}\partial_2 = \operatorname{Ker}\partial_1$, which as we saw was free abelian on three generators. In the present example, $H_1(X_2)$ is free abelian on two generators, $b - c$ and $c - d$, expressing the geometric fact that by filling in the 2-cell A we have reduced the number of 'holes' in our space from three to two.

Suppose we enlarge X_2 to a space X_3 by attaching a second 2-cell B along the same cycle $a - b$. This gives a 2-dimensional chain group C_2 consisting of linear combinations of A and B, and the boundary homomorphism $\partial_2 : C_2 \to C_1$ sends both A and B to $a - b$. The homology group $H_1(X_3) = \operatorname{Ker}\partial_1 / \operatorname{Im}\partial_2$ is the same as for X_2, but now ∂_2 has a nontrivial kernel, the infinite cyclic group generated by $A - B$. We view $A - B$ as a 2-dimensional cycle, generating the homology group $H_2(X_3) = \operatorname{Ker}\partial_2 \approx \mathbb{Z}$. Topologically, the cycle $A - B$ is the sphere formed by the cells A and B together with their common boundary circle. This spherical cycle detects the presence of a 'hole' in X_3, the missing interior of the sphere. However, since this hole is enclosed by a sphere rather than a circle, it is of a different sort from the holes detected by $H_1(X_3) \approx \mathbb{Z} \times \mathbb{Z}$, which are detected by the cycles $b - c$ and $c - d$.



Let us continue one more step and construct a complex X_4 from X_3 by attaching a 3-cell C along the 2-sphere formed by A and B. This creates a chain group C_3

generated by this 3-cell C, and we define a boundary homomorphism $\partial_3 : C_3 \to C_2$ sending C to $A - B$ since the cycle $A - B$ should be viewed as the boundary of C in the same way that the 1-dimensional cycle $a - b$ is the boundary of A. Now we have a sequence of three boundary homomorphisms $C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$ and the quotient $H_2(X_4) = \operatorname{Ker}\partial_2 / \operatorname{Im}\partial_3$ has become trivial. Also $H_3(X_4) = \operatorname{Ker}\partial_3 = 0$. The group $H_1(X_4)$ is the same as $H_1(X_3)$, namely $\mathbb{Z}\times\mathbb{Z}$, so this is the only nontrivial homology group of X_4.

It is clear what the general pattern of the examples is. For a cell complex X one has chain groups $C_n(X)$ which are free abelian groups with basis the n-cells of X, and there are boundary homomorphisms $\partial_n : C_n(X) \to C_{n-1}(X)$, in terms of which one defines the homology group $H_n(X) = \operatorname{Ker}\partial_n / \operatorname{Im}\partial_{n+1}$. The major difficulty is how to define ∂_n in general. For $n = 1$ this is easy: The boundary of an oriented edge is the vertex at its head minus the vertex at its tail. The next case $n = 2$ is also not hard, at least for cells attached along cycles that are simply loops of edges, for then the boundary of the cell is this cycle of edges, with the appropriate signs taking orientations into account. But for larger n, matters become more complicated. Even if one restricts attention to cell complexes formed from polyhedral cells with nice attaching maps, there is still the matter of orientations to sort out.

The best solution to this problem seems to be to adopt an indirect approach. Arbitrary polyhedra can always be subdivided into special polyhedra called simplices (the triangle and the tetrahedron are the 2-dimensional and 3-dimensional instances) so there is no loss of generality, though initially there is some loss of efficiency, in restricting attention entirely to simplices. For simplices there is no difficulty in defining boundary maps or in handling orientations. So one obtains a homology theory, called simplicial homology, for cell complexes built from simplices. Still, this is a rather restricted class of spaces, and the theory itself has a certain rigidity that makes it awkward to work with.

The way around these obstacles is to step back from the geometry of spaces decomposed into simplices and to consider instead something which at first glance seems wildly more complicated, the collection of all possible continuous maps of simplices into a given space X. These maps generate tremendously large chain groups $C_n(X)$, but the quotients $H_n(X) = \operatorname{Ker}\partial_n / \operatorname{Im}\partial_{n+1}$, called singular homology groups, turn out to be much smaller, at least for reasonably nice spaces X. In particular, for spaces like those in the four examples above, the singular homology groups coincide with the homology groups we computed from the cellular chains. And as we shall see later in this chapter, singular homology allows one to define these nice cellular homology groups for all cell complexes, and in particular to solve the problem of defining the boundary maps for cellular chains.

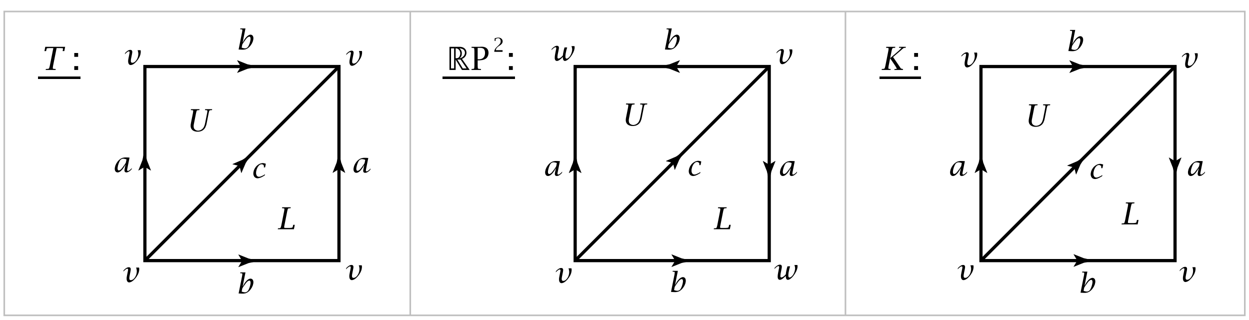
2.1 Simplicial and Singular Homology

The most important homology theory in algebraic topology, and the one we shall be studying almost exclusively, is called singular homology. Since the technical apparatus of singular homology is somewhat complicated, we will first introduce a more primitive version called simplicial homology in order to see how some of the apparatus works in a simpler setting before beginning the general theory.

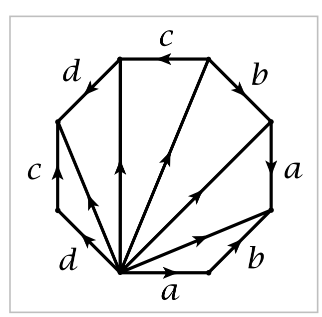
The natural domain of definition for simplicial homology is a class of spaces we call Δ-complexes, which are a mild generalization of the more classical notion of a simplicial complex. Historically, the modern definition of singular homology was first given in [Eilenberg 1944], and Δ-complexes were introduced soon thereafter in [Eilenberg-Zilber 1950] where they were called semisimplicial complexes. Within a few years this term came to be applied to what Eilenberg and Zilber called complete semisimplicial complexes, and later there was yet another shift in terminology as the latter objects came to be called simplicial sets. In theory this frees up the term semisimplicial complex to have its original meaning, but to avoid potential confusion it seems best to introduce a new name, and the term Δ-complex has at least the virtue of brevity.

Δ-Complexes

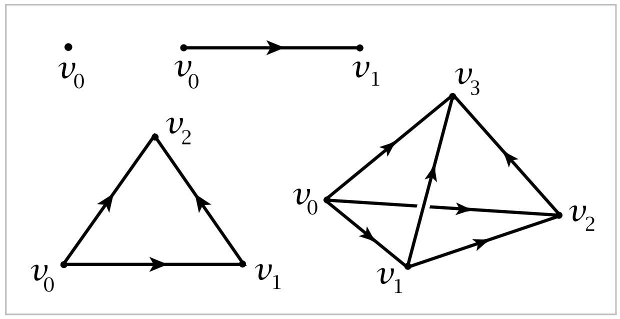
The torus, the projective plane, and the Klein bottle can each be obtained from a square by identifying opposite edges in the way indicated by the arrows in the following figures:



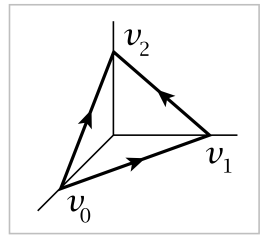
Cutting a square along a diagonal produces two triangles, so each of these surfaces can also be built from two triangles by identifying their edges in pairs. In similar fashion a polygon with any number of sides can be cut along diagonals into triangles, so in fact all closed surfaces can be constructed from triangles by identifying edges. Thus we have a single building block, the triangle, from which all surfaces can be constructed. Using only triangles we could also construct a large class of 2-dimensional spaces that are not surfaces in the strict sense, by allowing more than two edges to be identified together at a time.



The idea of a Δ-complex is to generalize constructions like these to any number of dimensions. The n-dimensional analog of the triangle is the ***n*-simplex**. This is the smallest convex set in a Euclidean space $\mathbb{R}^m$ containing $n+1$ points $v_0, \cdots, v_n$ that do not lie in a hyperplane of dimension less than n, where by a hyperplane we mean the set of solutions of a system of linear equations. An equivalent condition would be that the difference vectors $v_1 - v_0, \cdots, v_n - v_0$ are linearly independent. The points v_i are the **vertices** of the simplex, and the simplex itself is denoted $[v_0, \cdots, v_n]$. For example, there is the standard n-simplex



$$\Delta^n = \{ (t_0, \cdots, t_n) \in \mathbb{R}^{n+1} \mid \textstyle\sum_i t_i = 1 \text{ and } t_i \ge 0 \text{ for all } i \}$$



whose vertices are the unit vectors along the coordinate axes.

For purposes of homology it will be important to keep track of the order of the vertices of a simplex, so 'n-simplex' will really mean 'n-simplex with an ordering of its vertices'. A by-product of ordering the vertices of a simplex $[v_0, \cdots, v_n]$ is that this determines orientations of the edges $[v_i, v_j]$ according to increasing subscripts, as shown in the two preceding figures. Specifying the ordering of the vertices also determines a canonical linear homeomorphism from the standard n-simplex Δ^n onto any other n-simplex $[v_0, \cdots, v_n]$, preserving the order of vertices, namely, $(t_0, \cdots, t_n) \mapsto \sum_i t_i v_i$. The coefficients t_i are the **barycentric coordinates** of the point $\sum_i t_i v_i$ in $[v_0, \cdots, v_n]$.

If we delete one of the $n+1$ vertices of an n-simplex $[v_0, \cdots, v_n]$, then the remaining n vertices span an $(n-1)$-simplex, called a **face** of $[v_0, \cdots, v_n]$. We adopt the following convention:

The vertices of a face, or of any subsimplex spanned by a subset of the vertices, will always be ordered according to their order in the larger simplex.

The union of all the faces of Δ^n is the **boundary** of Δ^n, written $\partial\Delta^n$. The **open simplex** $\mathring{\Delta}^n$ is $\Delta^n - \partial\Delta^n$, the interior of Δ^n.

A **Δ-complex** structure on a space X is a collection of maps $\sigma_\alpha : \Delta^n \to X$, with n depending on the index α, such that:

(i) The restriction $\sigma_\alpha | \mathring{\Delta}^n$ is injective, and each point of X is in the image of exactly one such restriction $\sigma_\alpha | \mathring{\Delta}^n$.

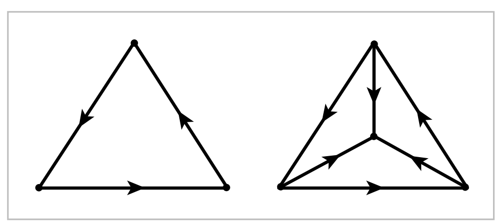
(ii) Each restriction of σ_α to a face of Δ^n is one of the maps $\sigma_\beta : \Delta^{n-1} \to X$. Here we are identifying the face of Δ^n with Δ^{n-1} by the canonical linear homeomorphism between them that preserves the ordering of the vertices.

(iii) A set $A \subset X$ is open iff $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each σ_α.

Among other things, this last condition rules out trivialities like regarding all the points of X as individual vertices. The earlier decompositions of the torus, projective plane, and Klein bottle into two triangles, three edges, and one or two vertices define Δ-complex structures with a total of six σ_α's for the torus and Klein bottle and seven for the projective plane. The orientations on the edges in the pictures are compatible with a unique ordering of the vertices of each simplex, and these orderings determine the maps σ_α.

A consequence of (iii) is that X can be built as a quotient space of a collection of disjoint simplices Δ^n_α, one for each $\sigma_\alpha : \Delta^n \to X$, the quotient space obtained by identifying each face of a Δ^n_α with the Δ^{n-1}_β corresponding to the restriction σ_β of σ_α to the face in question, as in condition (ii). One can think of building the quotient space inductively, starting with a discrete set of vertices, then attaching edges to these to produce a graph, then attaching 2-simplices to the graph, and so on. From this viewpoint we see that the data specifying a Δ-complex can be described purely combinatorially as collections of n-simplices Δ^n_α for each n together with functions associating to each face of each n-simplex Δ^n_α an $(n-1)$-simplex Δ^{n-1}_β.

More generally, Δ-complexes can be built from collections of disjoint simplices by identifying various subsimplices spanned by subsets of the vertices, where the identifications are performed using the canonical linear homeomorphisms that preserve the orderings of the vertices. The earlier Δ-complex structures on a torus, projective plane, or Klein bottle can be obtained in this way, by identifying pairs of edges of two 2-simplices. If one starts with a single 2-simplex and identifies all three edges to a single edge, preserving the orientations given by the ordering of the vertices, this produces a Δ-complex known as the 'dunce hat'. By contrast, if the three edges of a 2-simplex are identified preserving a cyclic orientation of the three edges, as in the first figure at the right, this does not produce a Δ-complex structure, although if the 2-simplex is subdivided into three smaller 2-simplices about a central vertex, then one does obtain a Δ-complex structure on the quotient space.

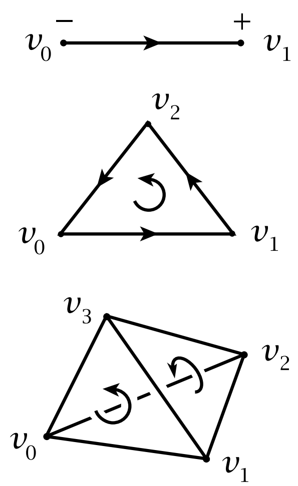


Thinking of a Δ-complex X as a quotient space of a collection of disjoint simplices, it is not hard to see that X must be a Hausdorff space. Condition (iii) then implies that each restriction $\sigma_\alpha | \mathring{\Delta}^n$ is a homeomorphism onto its image, which is thus an open simplex in X. It follows from Proposition A.2 in the Appendix that these open simplices $\sigma_\alpha(\mathring{\Delta}^n)$ are the cells e^n_α of a CW complex structure on X with the σ_α's as characteristic maps. We will not need this fact at present, however.

Simplicial Homology

Our goal now is to define the simplicial homology groups of a Δ-complex X. Let $\Delta_n(X)$ be the free abelian group with basis the open n-simplices e^n_α of X. Elements of $\Delta_n(X)$, called ***n*-chains**, can be written as finite formal sums $\sum_\alpha n_\alpha e_\alpha^n$ with coefficients $n_\alpha \in \mathbb{Z}$. Equivalently, we could write $\sum_\alpha n_\alpha \sigma_\alpha$ where $\sigma_\alpha : \Delta^n \to X$ is the characteristic map of e_α^n, with image the closure of e_α^n as described above. Such a sum $\sum_\alpha n_\alpha \sigma_\alpha$ can be thought of as a finite collection, or 'chain', of n-simplices in X with integer multiplicities, the coefficients n_α.

As one can see in the next figure, the boundary of the n-simplex $[v_0, \cdots, v_n]$ consists of the various $(n-1)$-dimensional simplices $[v_0, \cdots, \hat{v}_i, \cdots, v_n]$, where the 'hat' symbol $\hat{}$ over v_i indicates that this vertex is deleted from the sequence $v_0, \cdots, v_n$. In terms of chains, we might then wish to say that the boundary of $[v_0, \cdots, v_n]$ is the $(n-1)$-chain formed by the sum of the faces $[v_0, \cdots, \hat{v}_i, \cdots, v_n]$. However, it turns out to be better to insert certain signs and instead let the boundary of $[v_0, \cdots, v_n]$ be $\sum_i (-1)^i [v_0, \cdots, \hat{v}_i, \cdots, v_n]$. Heuristically, the signs are inserted to take orientations into account, so that all the faces of a simplex are coherently oriented, as indicated in the following figure:



$$\partial[v_0, v_1] = [v_1] - [v_0]$$

$$\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$

$$\partial[v_0, v_1, v_2, v_3] = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2]$$

In the last case, the orientations of the two hidden faces are also counterclockwise when viewed from outside the 3-simplex.

With this geometry in mind we define for a general Δ-complex X a **boundary homomorphism** $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ by specifying its values on basis elements:

$$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha | [v_0, \cdots, \hat{v}_i, \cdots, v_n]$$

Note that the right side of this equation does indeed lie in $\Delta_{n-1}(X)$ since each restriction $\sigma_\alpha | [v_0, \cdots, \hat{v}_i, \cdots, v_n]$ is the characteristic map of an $(n-1)$-simplex of X.

Lemma 2.1. *The composition* $\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X)$ *is zero.*

Proof: We have $\partial_n(\sigma) = \sum_i (-1)^i \sigma | [v_0, \cdots, \hat{v}_i, \cdots, v_n]$, and hence

$$\partial_{n-1}\partial_n(\sigma) = \sum_{j<i} (-1)^i (-1)^j \sigma | [v_0, \cdots, \hat{v}_j, \cdots, \hat{v}_i, \cdots, v_n] + \sum_{j>i} (-1)^i (-1)^{j-1} \sigma | [v_0, \cdots, \hat{v}_i, \cdots, \hat{v}_j, \cdots, v_n]$$

The latter two summations cancel since after switching i and j in the second sum, it becomes the negative of the first. $\square$

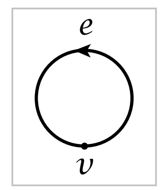
The algebraic situation we have now is a sequence of homomorphisms of abelian groups

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

with $\partial_n \partial_{n+1} = 0$ for each n. Such a sequence is called a **chain complex.** Note that we have extended the sequence by a 0 at the right end, with $\partial_0 = 0$. The equation $\partial_n \partial_{n+1} = 0$ is equivalent to the inclusion $\operatorname{Im} \partial_{n+1} \subset \operatorname{Ker} \partial_n$, where Im and Ker denote image and kernel. So we can define the n^{th} **homology group** of the chain complex to be the quotient group $H_n = \operatorname{Ker} \partial_n / \operatorname{Im} \partial_{n+1}$. Elements of $\operatorname{Ker} \partial_n$ are called **cycles** and elements of $\operatorname{Im} \partial_{n+1}$ are called **boundaries.** Elements of H_n are cosets of $\operatorname{Im} \partial_{n+1}$, called **homology classes.** Two cycles representing the same homology class are said to be **homologous.** This means their difference is a boundary.

Returning to the case that $C_n = \Delta_n(X)$, the homology group $\operatorname{Ker} \partial_n / \operatorname{Im} \partial_{n+1}$ will be denoted $H_n^\Delta(X)$ and called the n^{th} **simplicial homology group** of X.

Example 2.2. $X = S^1$, with one vertex v and one edge e. Then $\Delta_0(S^1)$ and $\Delta_1(S^1)$ are both $\mathbb{Z}$ and the boundary map ∂_1 is zero since $\partial e = v - v$. The groups $\Delta_n(S^1)$ are 0 for $n \geq 2$ since there are no simplices in these dimensions. Hence



$$H_n^\Delta(S^1) \approx \begin{cases} \mathbb{Z} & \text{for } n = 0, 1 \\ 0 & \text{for } n \geq 2 \end{cases}$$

This is an illustration of the general fact that if the boundary maps in a chain complex are all zero, then the homology groups of the complex are isomorphic to the chain groups themselves.

Example 2.3. $X = T$, the torus with the Δ-complex structure pictured earlier, having one vertex, three edges a, b, and c, and two 2-simplices U and L. As in the previous example, $\partial_1 = 0$ so $H_0^\Delta(T) \approx \mathbb{Z}$. Since $\partial_2 U = a + b - c = \partial_2 L$ and $\{a, b, a + b - c\}$ is a basis for $\Delta_1(T)$, it follows that $H_1^\Delta(T) \approx \mathbb{Z} \oplus \mathbb{Z}$ with basis the homology classes $[a]$ and $[b]$. Since there are no 3-simplices, $H_2^\Delta(T)$ is equal to $\operatorname{Ker} \partial_2$, which is infinite cyclic generated by $U - L$ since $\partial(pU + qL) = (p + q)(a + b - c) = 0$ only if $p = -q$. Thus

$$H_n^\Delta(T) \approx \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{for } n = 1 \\ \mathbb{Z} & \text{for } n = 0, 2 \\ 0 & \text{for } n \geq 3 \end{cases}$$

Example 2.4. $X = \mathbb{R}\mathrm{P}^2$, as pictured earlier, with two vertices v and w, three edges a, b, and c, and two 2-simplices U and L. Then $\operatorname{Im} \partial_1$ is generated by $w - v$, so $H_0^\Delta(X) \approx \mathbb{Z}$ with either vertex as a generator. Since $\partial_2 U = -a + b + c$ and $\partial_2 L = a - b + c$, we see that ∂_2 is injective, so $H_2^\Delta(X) = 0$. Further, $\operatorname{Ker} \partial_1 \approx \mathbb{Z} \oplus \mathbb{Z}$ with basis $a - b$ and c, and $\operatorname{Im} \partial_2$ is an index-two subgroup of $\operatorname{Ker} \partial_1$ since we can choose c and $a - b + c$

as a basis for $\operatorname{Ker}\partial_1$ and $a-b+c$ and $2c=(a-b+c)+(-a+b+c)$ as a basis for $\operatorname{Im}\partial_2$. Thus $H_1^{\Delta}(X)\approx\mathbb{Z}_2$.

Example 2.5. We can obtain a Δ-complex structure on S^n by taking two copies of Δ^n and identifying their boundaries via the identity map. Labeling these two n-simplices U and L, then it is obvious that $\operatorname{Ker}\partial_n$ is infinite cyclic generated by $U-L$. Thus $H_n^{\Delta}(S^n)\approx\mathbb{Z}$ for this Δ-complex structure on S^n. Computing the other homology groups would be more difficult.

Many similar examples could be worked out without much trouble, such as the other closed orientable and nonorientable surfaces. However, the calculations do tend to increase in complexity before long, particularly for higher-dimensional complexes.

Some obvious general questions arise: Are the groups $H_n^{\Delta}(X)$ independent of the choice of Δ-complex structure on X? In other words, if two Δ-complexes are homeomorphic, do they have isomorphic homology groups? More generally, do they have isomorphic homology groups if they are merely homotopy equivalent? To answer such questions and to develop a general theory it is best to leave the rather rigid simplicial realm and introduce the singular homology groups. These have the added advantage that they are defined for all spaces, not just Δ-complexes. At the end of this section, after some theory has been developed, we will show that simplicial and singular homology groups coincide for Δ-complexes.

Traditionally, simplicial homology is defined for **simplicial complexes**, which are the Δ-complexes whose simplices are uniquely determined by their vertices. This amounts to saying that each n-simplex has $n+1$ distinct vertices, and that no other n-simplex has this same set of vertices. Thus a simplicial complex can be described combinatorially as a set X_0 of vertices together with sets X_n of n-simplices, which are $(n+1)$-element subsets of X_0. The only requirement is that each $(k+1)$-element subset of the vertices of an n-simplex in X_n is a k-simplex, in X_k. From this combinatorial data a Δ-complex X can be constructed, once we choose a partial ordering of the vertices X_0 that restricts to a linear ordering on the vertices of each simplex in X_n. For example, we could just choose a linear ordering of all the vertices. This might perhaps involve invoking the Axiom of Choice for large vertex sets.

An exercise at the end of this section is to show that every Δ-complex can be subdivided to be a simplicial complex. In particular, every Δ-complex is then homeomorphic to a simplicial complex.

Compared with simplicial complexes, Δ-complexes have the advantage of simpler computations since fewer simplices are required. For example, to put a simplicial complex structure on the torus one needs at least 14 triangles, 21 edges, and 7 vertices, and for $\mathbb{R}\mathrm{P}^2$ one needs at least 10 triangles, 15 edges, and 6 vertices. This would slow down calculations considerably!

Singular Homology

A **singular n-simplex** in a space X is by definition just a map $\sigma:\Delta^n\to X$. The word 'singular' is used here to express the idea that σ need not be a nice embedding but can have 'singularities' where its image does not look at all like a simplex. All that is required is that σ be continuous. Let $C_n(X)$ be the free abelian group with basis the set of singular n-simplices in X. Elements of $C_n(X)$, called **n-chains**, or more precisely singular n-chains, are finite formal sums $\sum_i n_i\sigma_i$ for $n_i\in\mathbb{Z}$ and $\sigma_i:\Delta^n\to X$. A boundary map $\partial_n:C_n(X)\to C_{n-1}(X)$ is defined by the same formula as before:

$$\partial_n(\sigma)=\sum_i(-1)^i\sigma\,|\,[v_0,\cdots,\hat{v}_i,\cdots,v_n]$$

Implicit in this formula is the canonical identification of $[v_0,\cdots,\hat{v}_i,\cdots,v_n]$ with Δ^{n-1}, preserving the ordering of vertices, so that $\sigma\,|\,[v_0,\cdots,\hat{v}_i,\cdots,v_n]$ is regarded as a map $\Delta^{n-1}\to X$, that is, a singular $(n-1)$-simplex.

Often we write the boundary map ∂_n from $C_n(X)$ to $C_{n-1}(X)$ simply as ∂ when this does not lead to serious ambiguities. The proof of Lemma 2.1 applies equally well to singular simplices, showing that $\partial_n\partial_{n+1}=0$ or more concisely $\partial^2=0$, so we can define the **singular homology group** $H_n(X)=\operatorname{Ker}\partial_n/\operatorname{Im}\partial_{n+1}$.

It is evident from the definition that homeomorphic spaces have isomorphic singular homology groups H_n, in contrast with the situation for H_n^Δ. On the other hand, since the groups $C_n(X)$ are so large, the number of singular n-simplices in X usually being uncountable, it is not at all clear that for a Δ-complex X with finitely many simplices, $H_n(X)$ should be finitely generated for all n, or that $H_n(X)$ should be zero for n larger than the dimension of X — two properties that are trivial for $H_n^\Delta(X)$.

Though singular homology looks so much more general than simplicial homology, it can actually be regarded as a special case of simplicial homology by means of the following construction. For an arbitrary space X, define the **singular complex** $S(X)$ to be the Δ-complex with one n-simplex Δ_σ^n for each singular n-simplex $\sigma:\Delta^n\to X$, with Δ_σ^n attached in the obvious way to the $(n-1)$-simplices of $S(X)$ that are the restrictions of σ to the various $(n-1)$-simplices in $\partial\Delta^n$. It is clear from the definitions that $H_n^\Delta(S(X))$ is identical with $H_n(X)$ for all n, and in this sense the singular homology group $H_n(X)$ is a special case of a simplicial homology group. One can regard $S(X)$ as a Δ-complex model for X, although it is usually an extremely large object compared to X.

Cycles in singular homology are defined algebraically, but they can be given a somewhat more geometric interpretation in terms of maps from finite Δ-complexes. To see this, note first that a singular n-chain ξ can always be written in the form $\sum_i\varepsilon_i\sigma_i$ with $\varepsilon_i=\pm1$, allowing repetitions of the singular n-simplices σ_i. Given such an n-chain $\xi=\sum_i\varepsilon_i\sigma_i$, when we compute $\partial\xi$ as a sum of singular $(n-1)$-simplices with signs ±1, there may be some *canceling pairs* consisting of two identical singular $(n-1)$-simplices with opposite signs. Choosing a maximal collection of such canceling pairs, construct an n-dimensional Δ-complex K_ξ from a disjoint union of n-simplices Δ_i^n, one for each σ_i, by identifying the pairs of $(n-1)$-dimensional faces corresponding to the chosen canceling pairs. The σ_i's then induce a map $K_\xi \to X$. If ξ is a cycle, all the $(n-1)$-dimensional faces of the Δ_i^n's are identified in pairs. Thus K_ξ is a manifold, locally homeomorphic to $\mathbb{R}^n$, near all points in the complement of the $(n-2)$-skeleton K_ξ^{n-2} of K_ξ. All the n-simplices of K_ξ can be coherently oriented by taking the signs of the σ_i's into account, so $K_\xi - K_\xi^{n-2}$ is actually an oriented manifold. A closer inspection shows that K_ξ is also a manifold near points in the interiors of $(n-2)$-simplices, so the nonmanifold points of K_ξ in fact lie in the $(n-3)$-skeleton. However, near points in the interiors of $(n-3)$-simplices it can very well happen that K_ξ is not a manifold.

In particular, elements of $H_1(X)$ are represented by collections of oriented loops in X, and elements of $H_2(X)$ are represented by maps of closed oriented surfaces into X. With a bit more work it can be shown that an oriented 1-cycle $\coprod_\alpha S_\alpha^1 \to X$ is zero in $H_1(X)$ iff it extends to a map of a compact oriented surface with boundary $\coprod_\alpha S_\alpha^1$ into X. The analogous statement for 2-cycles is also true. In the early days of homology theory it may have been believed, or at least hoped, that this close connection with manifolds continued in all higher dimensions, but this has turned out not to be the case. There is a sort of homology theory built from manifolds, called *bordism*, but it is quite a bit more complicated than the homology theory we are studying here.

After these preliminary remarks let us begin to see what can be proved about singular homology.

Proposition 2.6. *Corresponding to the decomposition of a space X into its path-components X_α there is an isomorphism of $H_n(X)$ with the direct sum $\bigoplus_\alpha H_n(X_\alpha)$.*

Proof: Since a singular simplex always has path-connected image, $C_n(X)$ splits as the direct sum of its subgroups $C_n(X_\alpha)$. The boundary maps ∂_n preserve this direct sum decomposition, taking $C_n(X_\alpha)$ to $C_{n-1}(X_\alpha)$, so $\operatorname{Ker}\partial_n$ and $\operatorname{Im}\partial_{n+1}$ split similarly as direct sums, hence the homology groups also split, $H_n(X) \approx \bigoplus_\alpha H_n(X_\alpha)$. □

Proposition 2.7. *If X is nonempty and path-connected, then $H_0(X) \approx \mathbb{Z}$. Hence for any space X, $H_0(X)$ is a direct sum of $\mathbb{Z}$'s, one for each path-component of X.*

Proof: By definition, $H_0(X) = C_0(X)/\operatorname{Im}\partial_1$ since $\partial_0 = 0$. Define a homomorphism $\varepsilon : C_0(X) \to \mathbb{Z}$ by $\varepsilon(\sum_i n_i\sigma_i) = \sum_i n_i$. This is obviously surjective if X is nonempty. The claim is that $\operatorname{Ker}\varepsilon = \operatorname{Im}\partial_1$ if X is path-connected, and hence ε induces an isomorphism $H_0(X) \approx \mathbb{Z}$.

To verify the claim, observe first that $\operatorname{Im}\partial_1 \subset \operatorname{Ker}\varepsilon$ since for a singular 1-simplex $\sigma : \Delta^1 \to X$ we have $\varepsilon\partial_1(\sigma) = \varepsilon(\sigma|[v_1] - \sigma|[v_0]) = 1 - 1 = 0$. For the reverse inclusion $\operatorname{Ker}\varepsilon \subset \operatorname{Im}\partial_1$, suppose $\varepsilon(\sum_i n_i\sigma_i) = 0$, so $\sum_i n_i = 0$. The σ_i's are singular 0-simplices, which are simply points of X. Choose a path $\tau_i : I \to X$ from a basepoint

x_0 to $\sigma_i(v_0)$ and let σ_0 be the singular 0-simplex with image x_0. We can view τ_i as a singular 1-simplex, a map $\tau_i:[v_0,v_1]\to X$, and then we have $\partial\tau_i = \sigma_i - \sigma_0$. Hence $\partial(\sum_i n_i\tau_i) = \sum_i n_i\sigma_i - \sum_i n_i\sigma_0 = \sum_i n_i\sigma_i$ since $\sum_i n_i = 0$. Thus $\sum_i n_i\sigma_i$ is a boundary, which shows that $\operatorname{Ker}\varepsilon \subset \operatorname{Im}\partial_1$. □

Proposition 2.8. *If X is a point, then $H_n(X) = 0$ for $n > 0$ and $H_0(X) \approx \mathbb{Z}$.*

Proof: In this case there is a unique singular n-simplex σ_n for each n, and $\partial(\sigma_n) = \sum_i(-1)^i\sigma_{n-1}$, a sum of $n+1$ terms, which is therefore 0 for n odd and σ_{n-1} for n even, $n \neq 0$. Thus we have the chain complex

$$\cdots \longrightarrow \mathbb{Z} \xrightarrow{\approx} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\approx} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

with boundary maps alternately isomorphisms and trivial maps, except at the last $\mathbb{Z}$. The homology groups of this complex are trivial except for $H_0 \approx \mathbb{Z}$. □

It is often very convenient to have a slightly modified version of homology for which a point has trivial homology groups in all dimensions, including zero. This is done by defining the **reduced homology groups** $\widetilde{H}_n(X)$ to be the homology groups of the augmented chain complex

$$\cdots \longrightarrow C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

where $\varepsilon(\sum_i n_i\sigma_i) = \sum_i n_i$ as in the proof of Proposition 2.7. Here we had better require X to be nonempty, to avoid having a nontrivial homology group in dimension -1. Since $\varepsilon\partial_1 = 0$, ε vanishes on $\operatorname{Im}\partial_1$ and hence induces a map $H_0(X)\to\mathbb{Z}$ with kernel $\widetilde{H}_0(X)$, so $H_0(X) \approx \widetilde{H}_0(X)\oplus\mathbb{Z}$. Obviously $H_n(X) \approx \widetilde{H}_n(X)$ for $n > 0$.

Formally, one can think of the extra $\mathbb{Z}$ in the augmented chain complex as generated by the unique map $[\varnothing]\to X$ where $[\varnothing]$ is the empty simplex, with no vertices. The augmentation map ε is then the usual boundary map since $\partial[v_0] = [\hat{v}_0] = [\varnothing]$.

Readers who know about the fundamental group $\pi_1(X)$ may wish to make a detour here to look at §2.A where it is shown that $H_1(X)$ is the abelianization of $\pi_1(X)$ whenever X is path-connected. This result will not be needed elsewhere in the chapter, however.

Homotopy Invariance

The first substantial result we will prove about singular homology is that homotopy equivalent spaces have isomorphic homology groups. This will be done by showing that a map $f:X\to Y$ induces a homomorphism $f_*:H_n(X)\to H_n(Y)$ for each n, and that f_* is an isomorphism if f is a homotopy equivalence.

For a map $f:X\to Y$, an induced homomorphism $f_\sharp:C_n(X)\to C_n(Y)$ is defined by composing each singular n-simplex $\sigma:\Delta^n\to X$ with f to get a singular n-simplex

$f_\sharp(\sigma) = f\sigma : \Delta^n \to Y$, then extending $f_\sharp$ linearly via $f_\sharp(\sum_i n_i\sigma_i) = \sum_i n_i f_\sharp(\sigma_i) = \sum_i n_i f\sigma_i$. The maps $f_\sharp : C_n(X) \to C_n(Y)$ satisfy $f_\sharp\partial = \partial f_\sharp$ since

$$\begin{aligned} f_\sharp\partial(\sigma) &= f_\sharp(\textstyle\sum_i(-1)^i\sigma|[v_0,\cdots,\hat{v}_i,\cdots,v_n]) \\ &= \textstyle\sum_i(-1)^i f\sigma|[v_0,\cdots,\hat{v}_i,\cdots,v_n] = \partial f_\sharp(\sigma) \end{aligned}$$

Thus we have a diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C_{n+1}(X) & \xrightarrow{\partial} & C_n(X) & \xrightarrow{\partial} & C_{n-1}(X) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f_\sharp} & & \downarrow{\scriptstyle f_\sharp} & & \downarrow{\scriptstyle f_\sharp} & & \\ \cdots & \longrightarrow & C_{n+1}(Y) & \xrightarrow{\partial} & C_n(Y) & \xrightarrow{\partial} & C_{n-1}(Y) & \longrightarrow & \cdots \end{array}$$

such that in each square the composition $f_\sharp\partial$ equals the composition $\partial f_\sharp$. A diagram of maps with the property that any two compositions of maps starting at one point in the diagram and ending at another are equal is called a **commutative diagram.** In the present case commutativity of the diagram is equivalent to the commutativity relation $f_\sharp\partial = \partial f_\sharp$, but commutative diagrams can contain commutative triangles, pentagons, etc., as well as commutative squares.

The fact that the maps $f_\sharp : C_n(X) \to C_n(Y)$ satisfy $f_\sharp\partial = \partial f_\sharp$ is also expressed by saying that the $f_\sharp$'s define a **chain map** from the singular chain complex of X to that of Y. The relation $f_\sharp\partial = \partial f_\sharp$ implies that $f_\sharp$ takes cycles to cycles since $\partial\alpha = 0$ implies $\partial(f_\sharp\alpha) = f_\sharp(\partial\alpha) = 0$. Also, $f_\sharp$ takes boundaries to boundaries since $f_\sharp(\partial\beta) = \partial(f_\sharp\beta)$. Hence $f_\sharp$ induces a homomorphism $f_* : H_n(X) \to H_n(Y)$. An algebraic statement of what we have just proved is:

Proposition 2.9. *A chain map between chain complexes induces homomorphisms between the homology groups of the two complexes.* □

Two basic properties of induced homomorphisms which are important in spite of being rather trivial are:

(i) $(fg)_* = f_*g_*$ for a composed mapping $X \xrightarrow{g} Y \xrightarrow{f} Z$. This follows from associativity of compositions $\Delta^n \xrightarrow{\sigma} X \xrightarrow{g} Y \xrightarrow{f} Z$.

(ii) $\mathbb{1}_* = \mathbb{1}$ where $\mathbb{1}$ denotes the identity map of a space or a group.

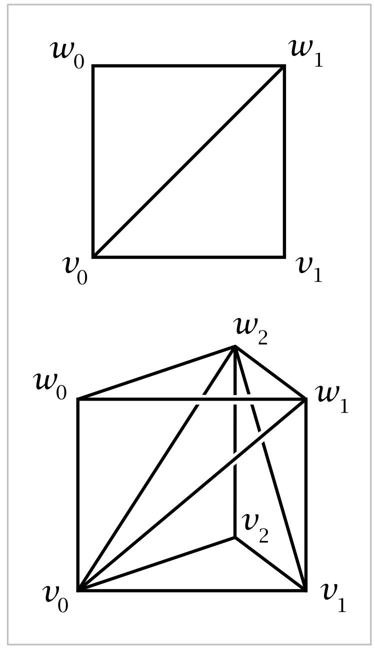
Less trivially, we have:

Theorem 2.10. *If two maps $f, g : X \to Y$ are homotopic, then they induce the same homomorphism $f_* = g_* : H_n(X) \to H_n(Y)$.*

In view of the formal properties $(fg)_* = f_*g_*$ and $\mathbb{1}_* = \mathbb{1}$, this immediately implies:

Corollary 2.11. *The maps $f_* : H_n(X) \to H_n(Y)$ induced by a homotopy equivalence $f : X \to Y$ are isomorphisms for all n.* □

For example, if X is contractible then $\widetilde{H}_n(X) = 0$ for all n.



Proof of 2.10: The essential ingredient is a procedure for subdividing $\Delta^n \times I$ into simplices. The figure shows the cases $n = 1, 2$. In $\Delta^n \times I$, let $\Delta^n \times \{0\} = [v_0, \cdots, v_n]$ and $\Delta^n \times \{1\} = [w_0, \cdots, w_n]$, where v_i and w_i have the same image under the projection $\Delta^n \times I \to \Delta^n$. We can pass from $[v_0, \cdots, v_n]$ to $[w_0, \cdots, w_n]$ by interpolating a sequence of n-simplices, each obtained from the preceding one by moving one vertex v_i up to w_i, starting with v_n and working backwards to v_0. Thus the first step is to move $[v_0, \cdots, v_n]$ up to $[v_0, \cdots, v_{n-1}, w_n]$, then the second step is to move this up to $[v_0, \cdots, v_{n-2}, w_{n-1}, w_n]$, and so on. In the typical step $[v_0, \cdots, v_i, w_{i+1}, \cdots, w_n]$ moves up to $[v_0, \cdots, v_{i-1}, w_i, \cdots, w_n]$. The region between these two n-simplices is exactly the $(n+1)$-simplex $[v_0, \cdots, v_i, w_i, \cdots, w_n]$ which has $[v_0, \cdots, v_i, w_{i+1}, \cdots, w_n]$ as its lower face and $[v_0, \cdots, v_{i-1}, w_i, \cdots, w_n]$ as its upper face. Altogether, $\Delta^n \times I$ is the union of the $(n+1)$-simplices $[v_0, \cdots, v_i, w_i, \cdots, w_n]$, each intersecting the next in an n-simplex face.

Given a homotopy $F: X \times I \to Y$ from f to g and a singular simplex $\sigma: \Delta^n \to X$, we can form the composition $F \circ (\sigma \times \mathbb{1}): \Delta^n \times I \to X \times I \to Y$. Using this, we can define *prism operators* $P: C_n(X) \to C_{n+1}(Y)$ by the following formula:

$$P(\sigma) = \sum_i (-1)^i F \circ (\sigma \times \mathbb{1}) \,|\, [v_0, \cdots, v_i, w_i, \cdots, w_n]$$

We will show that these prism operators satisfy the basic relation

$$\partial P = g_\sharp - f_\sharp - P\partial$$

Geometrically, the left side of this equation represents the boundary of the prism, and the three terms on the right side represent the top $\Delta^n \times \{1\}$, the bottom $\Delta^n \times \{0\}$, and the sides $\partial \Delta^n \times I$ of the prism. To prove the relation we calculate

$$\begin{aligned} \partial P(\sigma) = & \sum_{j \le i} (-1)^i (-1)^j F \circ (\sigma \times \mathbb{1}) \,|\, [v_0, \cdots, \hat{v}_j, \cdots, v_i, w_i, \cdots, w_n] \\ & + \sum_{j \ge i} (-1)^i (-1)^{j+1} F \circ (\sigma \times \mathbb{1}) \,|\, [v_0, \cdots, v_i, w_i, \cdots, \widehat{w}_j, \cdots, w_n] \end{aligned}$$

The terms with $i = j$ in the two sums cancel except for $F \circ (\sigma \times \mathbb{1}) \,|\, [\hat{v}_0, w_0, \cdots, w_n]$, which is $g \circ \sigma = g_\sharp(\sigma)$, and $-F \circ (\sigma \times \mathbb{1}) \,|\, [v_0, \cdots, v_n, \widehat{w}_n]$, which is $-f \circ \sigma = -f_\sharp(\sigma)$. The terms with $i \ne j$ are exactly $-P\partial(\sigma)$ since

$$\begin{aligned} P\partial(\sigma) = & \sum_{i < j} (-1)^i (-1)^j F \circ (\sigma \times \mathbb{1}) \,|\, [v_0, \cdots, v_i, w_i, \cdots, \widehat{w}_j, \cdots, w_n] \\ & + \sum_{i > j} (-1)^{i-1} (-1)^j F \circ (\sigma \times \mathbb{1}) \,|\, [v_0, \cdots, \hat{v}_j, \cdots, v_i, w_i, \cdots, w_n] \end{aligned}$$

Now we can finish the proof of the theorem. If $\alpha \in C_n(X)$ is a cycle, then we have $g_\sharp(\alpha) - f_\sharp(\alpha) = \partial P(\alpha) + P\partial(\alpha) = \partial P(\alpha)$ since $\partial\alpha = 0$. Thus $g_\sharp(\alpha) - f_\sharp(\alpha)$ is a boundary, so $g_\sharp(\alpha)$ and $f_\sharp(\alpha)$ determine the same homology class, which means that g_* equals f_* on the homology class of α. □

The relationship $\partial P + P\partial = g_\sharp - f_\sharp$ is expressed by saying P is a **chain homotopy** between the chain maps $f_\sharp$ and $g_\sharp$. We have just shown:

Proposition 2.12. *Chain-homotopic chain maps induce the same homomorphism on homology.* □

There are also induced homomorphisms $f_* : \widetilde{H}_n(X) \to \widetilde{H}_n(Y)$ for reduced homology groups since $f_\sharp\varepsilon = \varepsilon f_\sharp$ where $f_\sharp$ is the identity map on the added groups $\mathbb{Z}$ in the augmented chain complexes. The properties of induced homomorphisms we proved above hold equally well in the setting of reduced homology, with the same proofs.

Exact Sequences and Excision

If there was always a simple relationship between the homology groups of a space X, a subspace A, and the quotient space X/A, then this could be a very useful tool in understanding the homology groups of spaces such as CW complexes that can be built inductively from successively more complicated subspaces. Perhaps the simplest possible relationship would be if $H_n(X)$ contained $H_n(A)$ as a subgroup and the quotient group $H_n(X)/H_n(A)$ was isomorphic to $H_n(X/A)$. While this does hold in some cases, if it held in general then homology theory would collapse totally since every space X can be embedded as a subspace of a space with trivial homology groups, namely the cone $CX = (X\times I)/(X\times\{0\})$, which is contractible.

It turns out that this overly simple model does not have to be modified too much to get a relationship that is valid in fair generality. The novel feature of the actual relationship is that it involves the groups $H_n(X)$, $H_n(A)$, and $H_n(X/A)$ for all values of n simultaneously. In practice this is not as bad as it might sound, and in addition it has the pleasant side effect of sometimes allowing higher-dimensional homology groups to be computed in terms of lower-dimensional groups which may already be known, for example by induction.

In order to formulate the relationship we are looking for, we need an algebraic definition which is central to algebraic topology. A sequence of homomorphisms

$$\cdots \longrightarrow A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \longrightarrow \cdots$$

is said to be **exact** if $\operatorname{Ker}\alpha_n = \operatorname{Im}\alpha_{n+1}$ for each n. The inclusions $\operatorname{Im}\alpha_{n+1} \subset \operatorname{Ker}\alpha_n$ are equivalent to $\alpha_n\alpha_{n+1} = 0$, so the sequence is a chain complex, and the opposite inclusions $\operatorname{Ker}\alpha_n \subset \operatorname{Im}\alpha_{n+1}$ say that the homology groups of this chain complex are trivial.

A number of basic algebraic concepts can be expressed in terms of exact sequences, for example:

(i) $0 \longrightarrow A \xrightarrow{\alpha} B$ is exact iff $\operatorname{Ker}\alpha = 0$, i.e., α is injective.
(ii) $A \xrightarrow{\alpha} B \longrightarrow 0$ is exact iff $\operatorname{Im}\alpha = B$, i.e., α is surjective.
(iii) $0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow 0$ is exact iff α is an isomorphism, by (i) and (ii).
(iv) $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ is exact iff α is injective, β is surjective, and $\operatorname{Ker}\beta = \operatorname{Im}\alpha$, so β induces an isomorphism $C \approx B/\operatorname{Im}\alpha$. This can be written $C \approx B/A$ if we think of α as an inclusion of A as a subgroup of B.

An exact sequence $0 \to A \to B \to C \to 0$ as in (iv) is called a **short exact sequence**.

Exact sequences provide the right tool to relate the homology groups of a space, a subspace, and the associated quotient space:

Theorem 2.13. *If X is a space and A is a nonempty closed subspace that is a deformation retract of some neighborhood in X, then there is an exact sequence*

$$\cdots \longrightarrow \widetilde{H}_n(A) \xrightarrow{i_*} \widetilde{H}_n(X) \xrightarrow{j_*} \widetilde{H}_n(X/A) \xrightarrow{\partial} \widetilde{H}_{n-1}(A) \xrightarrow{i_*} \widetilde{H}_{n-1}(X) \longrightarrow \cdots$$
$$\cdots \longrightarrow \widetilde{H}_0(X/A) \longrightarrow 0$$

where i is the inclusion $A \hookrightarrow X$ and j is the quotient map $X \to X/A$.

The map ∂ will be constructed in the course of the proof. The idea is that an element $x \in \widetilde{H}_n(X/A)$ can be represented by a chain α in X with $\partial\alpha$ a cycle in A whose homology class is $\partial x \in \widetilde{H}_{n-1}(A)$.

Pairs of spaces (X, A) satisfying the hypothesis of the theorem will be called **good pairs**. For example, if X is a CW complex and A is a nonempty subcomplex, then (X, A) is a good pair by Proposition A.5 in the Appendix.

Corollary 2.14. $\widetilde{H}_n(S^n) \approx \mathbb{Z}$ *and* $\widetilde{H}_i(S^n) = 0$ *for* $i \neq n$.

Proof: For $n > 0$ take $(X, A) = (D^n, S^{n-1})$ so $X/A = S^n$. The terms $\widetilde{H}_i(D^n)$ in the long exact sequence for this pair are zero since D^n is contractible. Exactness of the sequence then implies that the maps $\widetilde{H}_i(S^n) \xrightarrow{\partial} \widetilde{H}_{i-1}(S^{n-1})$ are isomorphisms for $i > 0$ and that $\widetilde{H}_0(S^n) = 0$. The result now follows by induction on n, starting with the case of S^0 where the result holds by Propositions 2.6 and 2.8. □

As an application of this calculation we have the following classical theorem of Brouwer, the 2-dimensional case of which was proved in §1.1.

Corollary 2.15. ∂D^n *is not a retract of* D^n*. Hence every map* $f : D^n \to D^n$ *has a fixed point.*

Proof: If $r : D^n \to \partial D^n$ is a retraction, then $ri = \mathbb{1}$ for $i : \partial D^n \to D^n$ the inclusion map. The composition $\widetilde{H}_{n-1}(\partial D^n) \xrightarrow{i_*} \widetilde{H}_{n-1}(D^n) \xrightarrow{r_*} \widetilde{H}_{n-1}(\partial D^n)$ is then the identity map on $\widetilde{H}_{n-1}(\partial D^n) \approx \mathbb{Z}$. But i_* and r_* are both 0 since $\widetilde{H}_{n-1}(D^n) = 0$, and we have a contradiction. The statement about fixed points follows as in Theorem 1.9. $\square$

The derivation of the exact sequence of homology groups for a good pair (X,A) will be rather a long story. We will in fact derive a more general exact sequence which holds for arbitrary pairs (X,A), but with the homology groups of the quotient space X/A replaced by *relative homology groups*, denoted $H_n(X,A)$. These turn out to be quite useful for many other purposes as well.

Relative Homology Groups

It sometimes happens that by ignoring a certain amount of data or structure one obtains a simpler, more flexible theory which, almost paradoxically, can give results not readily obtainable in the original setting. A familiar instance of this is arithmetic mod n, where one ignores multiples of n. Relative homology is another example. In this case what one ignores is all singular chains in a subspace of the given space.

Relative homology groups are defined in the following way. Given a space X and a subspace $A \subset X$, let $C_n(X,A)$ be the quotient group $C_n(X)/C_n(A)$. Thus chains in A are trivial in $C_n(X,A)$. Since the boundary map $\partial : C_n(X) \to C_{n-1}(X)$ takes $C_n(A)$ to $C_{n-1}(A)$, it induces a quotient boundary map $\partial : C_n(X,A) \to C_{n-1}(X,A)$. Letting n vary, we have a sequence of boundary maps

$$\cdots \longrightarrow C_n(X,A) \xrightarrow{\partial} C_{n-1}(X,A) \longrightarrow \cdots$$

The relation $\partial^2 = 0$ holds for these boundary maps since it holds before passing to quotient groups. So we have a chain complex, and the homology groups $\operatorname{Ker}\partial / \operatorname{Im}\partial$ of this chain complex are by definition the **relative homology groups** $H_n(X,A)$. By considering the definition of the relative boundary map we see:

- Elements of $H_n(X,A)$ are represented by **relative cycles**: n-chains $\alpha \in C_n(X)$ such that $\partial\alpha \in C_{n-1}(A)$.
- A relative cycle α is trivial in $H_n(X,A)$ iff it is a **relative boundary**: $\alpha = \partial\beta + \gamma$ for some $\beta \in C_{n+1}(X)$ and $\gamma \in C_n(A)$.

These properties make precise the intuitive idea that $H_n(X,A)$ is 'homology of X modulo A'.

The quotient $C_n(X)/C_n(A)$ could also be viewed as a subgroup of $C_n(X)$, the subgroup with basis the singular n-simplices $\sigma : \Delta^n \to X$ whose image is not contained in A. However, the boundary map does not take this subgroup of $C_n(X)$ to the corresponding subgroup of $C_{n-1}(X)$, so it is usually better to regard $C_n(X,A)$ as a quotient rather than a subgroup of $C_n(X)$.

Our goal now is to show that the relative homology groups $H_n(X,A)$ for any pair (X,A) fit into a long exact sequence

$$\cdots \longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X,A) \longrightarrow H_{n-1}(A) \longrightarrow H_{n-1}(X) \longrightarrow \cdots$$
$$\cdots \longrightarrow H_0(X,A) \longrightarrow 0$$

This will be entirely a matter of algebra. To start the process, consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & C_n(A) \arrow[r, "i"] \arrow[d, "\partial"] & C_n(X) \arrow[r, "j"] \arrow[d, "\partial"] & C_n(X,A) \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r, "i"] & C_{n-1}(X) \arrow[r, "j"] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}$$

where i is inclusion and j is the quotient map. The diagram is commutative by the definition of the boundary maps. Letting n vary, and drawing these short exact sequences vertically rather than horizontally, we have a large commutative diagram of the form shown at the right, where the columns are exact and the rows are chain complexes which we denote A, B, and C. Such a diagram is called a **short exact sequence of chain complexes**. We will show that when we pass to homology groups, this short exact sequence of chain complexes stretches out into a long exact sequence of homology groups

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A_{n+1} \arrow[r, "\partial"] \arrow[d, "i"] & A_n \arrow[r, "\partial"] \arrow[d, "i"] & A_{n-1} \arrow[r] \arrow[d, "i"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "\partial"] \arrow[d, "j"] & B_n \arrow[r, "\partial"] \arrow[d, "j"] & B_{n-1} \arrow[r] \arrow[d, "j"] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial"] \arrow[d] & C_n \arrow[r, "\partial"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

$$\cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \longrightarrow \cdots$$

where $H_n(A)$ denotes the homology group $\operatorname{Ker}\partial / \operatorname{Im}\partial$ at A_n in the chain complex A, and $H_n(B)$ and $H_n(C)$ are defined similarly.

The commutativity of the squares in the short exact sequence of chain complexes means that i and j are chain maps. These therefore induce maps i_* and j_* on homology. To define the boundary map $\partial : H_n(C) \to H_{n-1}(A)$, let $c \in C_n$ be a cycle. Since j is onto, $c = j(b)$ for some $b \in B_n$. The element $\partial b \in B_{n-1}$ is in $\operatorname{Ker} j$ since $j(\partial b) = \partial j(b) = \partial c = 0$. So $\partial b = i(a)$ for some $a \in A_{n-1}$ since $\operatorname{Ker} j = \operatorname{Im} i$. Note that $\partial a = 0$ since $i(\partial a) = \partial i(a) = \partial\partial b = 0$ and i is injective. We define $\partial : H_n(C) \to H_{n-1}(A)$ by sending the homology class of c to the homology class of a, $\partial[c] = [a]$. This is well-defined since:

$$\begin{tikzcd}
 & & a \arrow[d, mapsto] & A_{n-1} \arrow[dd, "i"] \\
b \arrow[rr, mapsto] \arrow[dd, mapsto] & & \partial b & \\
 & B_n \arrow[rr, "\partial"'] \arrow[d, "j"] & & B_{n-1} \\
c & C_n & &
\end{tikzcd}$$

- The element a is uniquely determined by ∂b since i is injective.
- A different choice b' for b would have $j(b') = j(b)$, so $b' - b$ is in $\operatorname{Ker} j = \operatorname{Im} i$. Thus $b' - b = i(a')$ for some a', hence $b' = b + i(a')$. The effect of replacing b by $b + i(a')$ is to change a to the homologous element $a + \partial a'$ since $i(a + \partial a') = i(a) + i(\partial a') = \partial b + \partial i(a') = \partial(b + i(a'))$.
- A different choice of c within its homology class would have the form $c + \partial c'$. Since $c' = j(b')$ for some b', we then have $c + \partial c' = c + \partial j(b') = c + j(\partial b') = j(b + \partial b')$, so b is replaced by $b + \partial b'$, which leaves ∂b and therefore also a unchanged.

The map $\partial : H_n(C) \to H_{n-1}(A)$ is a homomorphism since if $\partial[c_1] = [a_1]$ and $\partial[c_2] = [a_2]$ via elements b_1 and b_2 as above, then $j(b_1 + b_2) = j(b_1) + j(b_2) = c_1 + c_2$ and $i(a_1 + a_2) = i(a_1) + i(a_2) = \partial b_1 + \partial b_2 = \partial(b_1 + b_2)$, so $\partial([c_1] + [c_2]) = [a_1] + [a_2]$.

Theorem 2.16. *The sequence of homology groups*

$$\cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \longrightarrow \cdots$$

is exact.

Proof: There are six things to verify:

$\operatorname{Im} i_* \subset \operatorname{Ker} j_*$. This is immediate since $ji = 0$ implies $j_* i_* = 0$.

$\operatorname{Im} j_* \subset \operatorname{Ker} \partial$. We have $\partial j_* = 0$ since in this case $\partial b = 0$ in the definition of ∂.

$\operatorname{Im} \partial \subset \operatorname{Ker} i_*$. Here $i_* \partial = 0$ since $i_* \partial$ takes $[c]$ to $[\partial b] = 0$.

$\operatorname{Ker} j_* \subset \operatorname{Im} i_*$. A homology class in $\operatorname{Ker} j_*$ is represented by a cycle $b \in B_n$ with $j(b)$ a boundary, so $j(b) = \partial c'$ for some $c' \in C_{n+1}$. Since j is surjective, $c' = j(b')$ for some $b' \in B_{n+1}$. We have $j(b - \partial b') = j(b) - j(\partial b') = j(b) - \partial j(b') = 0$ since $\partial j(b') = \partial c' = j(b)$. So $b - \partial b' = i(a)$ for some $a \in A_n$. This a is a cycle since $i(\partial a) = \partial i(a) = \partial(b - \partial b') = \partial b = 0$ and i is injective. Thus $i_*[a] = [b - \partial b'] = [b]$, showing that i_* maps onto $\operatorname{Ker} j_*$.

$\operatorname{Ker} \partial \subset \operatorname{Im} j_*$. In the notation used in the definition of ∂, if c represents a homology class in $\operatorname{Ker} \partial$, then $a = \partial a'$ for some $a' \in A_n$. The element $b - i(a')$ is a cycle since $\partial(b - i(a')) = \partial b - \partial i(a') = \partial b - i(\partial a') = \partial b - i(a) = 0$. And $j(b - i(a')) = j(b) - ji(a') = j(b) = c$, so j_* maps $[b - i(a')]$ to $[c]$.

$\operatorname{Ker} i_* \subset \operatorname{Im} \partial$. Given a cycle $a \in A_{n-1}$ such that $i(a) = \partial b$ for some $b \in B_n$, then $j(b)$ is a cycle since $\partial j(b) = j(\partial b) = ji(a) = 0$, and ∂ takes $[j(b)]$ to $[a]$. $\square$

This theorem represents the beginnings of the subject of homological algebra. The method of proof is sometimes called *diagram chasing*.

Returning to topology, the preceding algebraic theorem yields a long exact sequence of homology groups:

$$\cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \longrightarrow \cdots$$
$$\cdots \longrightarrow H_0(X,A) \longrightarrow 0$$

The boundary map $\partial : H_n(X,A) \to H_{n-1}(A)$ has a very simple description: If a class $[\alpha] \in H_n(X,A)$ is represented by a relative cycle α, then $\partial[\alpha]$ is the class of the cycle $\partial\alpha$ in $H_{n-1}(A)$. This is immediate from the algebraic definition of the boundary homomorphism in the long exact sequence of homology groups associated to a short exact sequence of chain complexes.

This long exact sequence makes precise the idea that the groups $H_n(X,A)$ measure the difference between the groups $H_n(X)$ and $H_n(A)$. In particular, exactness implies that if $H_n(X,A)=0$ for all n, then the inclusion $A \hookrightarrow X$ induces isomorphisms $H_n(A) \approx H_n(X)$ for all n, by the remark (iii) following the definition of exactness. The converse is also true according to an exercise at the end of this section.

There is a completely analogous long exact sequence of reduced homology groups for a pair (X,A) with $A \neq \varnothing$. This comes from applying the preceding algebraic machinery to the short exact sequence of chain complexes formed by the short exact sequences $0 \to C_n(A) \to C_n(X) \to C_n(X,A) \to 0$ in nonnegative dimensions, augmented by the short exact sequence $0 \longrightarrow \mathbb{Z} \xrightarrow{\mathbb{1}} \mathbb{Z} \longrightarrow 0 \longrightarrow 0$ in dimension -1. In particular this means that $\widetilde{H}_n(X,A)$ is the same as $H_n(X,A)$ for all n, when $A \neq \varnothing$.

Example 2.17. In the long exact sequence of reduced homology groups for the pair $(D^n, \partial D^n)$, the maps $H_i(D^n, \partial D^n) \xrightarrow{\partial} \widetilde{H}_{i-1}(S^{n-1})$ are isomorphisms for all $i > 0$ since the remaining terms $\widetilde{H}_i(D^n)$ are zero for all i. Thus we obtain the calculation

$$H_i(D^n, \partial D^n) \approx \begin{cases} \mathbb{Z} & \text{for } i = n \\ 0 & \text{otherwise} \end{cases}$$

Example 2.18. Applying the long exact sequence of reduced homology groups to a pair (X, x_0) with $x_0 \in X$ yields isomorphisms $H_n(X, x_0) \approx \widetilde{H}_n(X)$ for all n since $\widetilde{H}_n(x_0) = 0$ for all n.

There are induced homomorphisms for relative homology just as there are in the nonrelative, or 'absolute', case. A map $f: X \to Y$ with $f(A) \subset B$, or more concisely $f:(X,A) \to (Y,B)$, induces homomorphisms $f_\sharp : C_n(X,A) \to C_n(Y,B)$ since the chain map $f_\sharp : C_n(X) \to C_n(Y)$ takes $C_n(A)$ to $C_n(B)$, so we get a well-defined map on quotients, $f_\sharp : C_n(X,A) \to C_n(Y,B)$. The relation $f_\sharp \partial = \partial f_\sharp$ holds for relative chains since it holds for absolute chains. By Proposition 2.9 we then have induced homomorphisms $f_* : H_n(X,A) \to H_n(Y,B)$.

Proposition 2.19. *If two maps $f, g : (X,A) \to (Y,B)$ are homotopic through maps of pairs $(X,A) \to (Y,B)$, then $f_* = g_* : H_n(X,A) \to H_n(Y,B)$.*

Proof: The prism operator P from the proof of Theorem 2.10 takes $C_n(A)$ to $C_{n+1}(B)$, hence induces a relative prism operator $P : C_n(X,A) \to C_{n+1}(Y,B)$. Since we are just passing to quotient groups, the formula $\partial P + P \partial = g_\sharp - f_\sharp$ remains valid. Thus the maps $f_\sharp$ and $g_\sharp$ on relative chain groups are chain homotopic, and hence they induce the same homomorphism on relative homology groups. ☐

An easy generalization of the long exact sequence of a pair (X,A) is the long exact sequence of a triple (X,A,B), where $B \subset A \subset X$:

$$\cdots \longrightarrow H_n(A,B) \longrightarrow H_n(X,B) \longrightarrow H_n(X,A) \longrightarrow H_{n-1}(A,B) \longrightarrow \cdots$$

This is the long exact sequence of homology groups associated to the short exact sequence of chain complexes formed by the short exact sequences

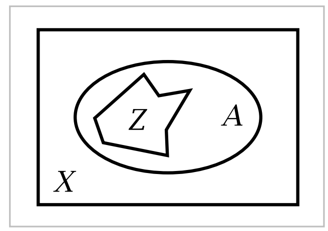
$$0 \longrightarrow C_n(A,B) \longrightarrow C_n(X,B) \longrightarrow C_n(X,A) \longrightarrow 0$$

For example, taking B to be a point, the long exact sequence of the triple (X, A, B) becomes the long exact sequence of reduced homology for the pair (X, A).

Excision

A fundamental property of relative homology groups is given by the following **Excision Theorem**, describing when the relative groups $H_n(X, A)$ are unaffected by deleting, or excising, a subset $Z \subset A$.

Theorem 2.20. *Given subspaces $Z \subset A \subset X$ such that the closure of Z is contained in the interior of A, then the inclusion $(X - Z, A - Z) \hookrightarrow (X, A)$ induces isomorphisms $H_n(X - Z, A - Z) \to H_n(X, A)$ for all n. Equivalently, for subspaces $A, B \subset X$ whose interiors cover X, the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms $H_n(B, A \cap B) \to H_n(X, A)$ for all n.*



The translation between the two versions is obtained by setting $B = X - Z$ and $Z = X - B$. Then $A \cap B = A - Z$ and the condition $\mathrm{cl}\, Z \subset \mathrm{int}\, A$ is equivalent to $X = \mathrm{int}\, A \cup \mathrm{int}\, B$ since $X - \mathrm{int}\, B = \mathrm{cl}\, Z$.

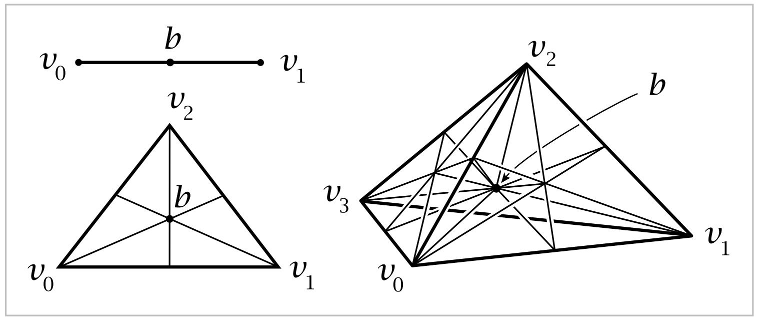
The proof of the excision theorem will involve a rather lengthy technical detour involving a construction known as barycentric subdivision, which allows homology groups to be computed using small singular simplices. In a metric space 'smallness' can be defined in terms of diameters, but for general spaces it will be defined in terms of covers.

For a space X, let $\mathcal{U} = \{U_j\}$ be a collection of subspaces of X whose interiors form an open cover of X, and let $C_n^{\mathcal{U}}(X)$ be the subgroup of $C_n(X)$ consisting of chains $\sum_i n_i \sigma_i$ such that each σ_i has image contained in some set in the cover $\mathcal{U}$. The boundary map $\partial : C_n(X) \to C_{n-1}(X)$ takes $C_n^{\mathcal{U}}(X)$ to $C_{n-1}^{\mathcal{U}}(X)$, so the groups $C_n^{\mathcal{U}}(X)$ form a chain complex. We denote the homology groups of this chain complex by $H_n^{\mathcal{U}}(X)$.

Proposition 2.21. *The inclusion $\iota : C_n^{\mathcal{U}}(X) \hookrightarrow C_n(X)$ is a chain homotopy equivalence, that is, there is a chain map $\rho : C_n(X) \to C_n^{\mathcal{U}}(X)$ such that $\iota\rho$ and $\rho\iota$ are chain homotopic to the identity. Hence ι induces isomorphisms $H_n^{\mathcal{U}}(X) \approx H_n(X)$ for all n.*

Proof: The barycentric subdivision process will be performed at four levels, beginning with the most geometric and becoming increasingly algebraic.

(1) *Barycentric Subdivision of Simplices.* The points of a simplex $[v_0, \cdots, v_n]$ are the linear combinations $\sum_i t_i v_i$ with $\sum_i t_i = 1$ and $t_i \ge 0$ for each i. The **barycenter** or 'center of gravity' of the simplex $[v_0, \cdots, v_n]$ is the point $b = \sum_i t_i v_i$ whose barycentric coordinates t_i are all equal, namely $t_i = 1/(n+1)$ for each i. The **barycentric subdivision** of $[v_0, \cdots, v_n]$ is the decomposition of $[v_0, \cdots, v_n]$ into the n-simplices $[b, w_0, \cdots, w_{n-1}]$ where, inductively, $[w_0, \cdots, w_{n-1}]$ is an $(n-1)$-simplex in the barycentric subdivision of a face $[v_0, \cdots, \hat{v}_i, \cdots, v_n]$. The induction starts with the case $n = 0$ when the barycentric subdivision of $[v_0]$ is defined to be just $[v_0]$ itself. The next two cases $n = 1, 2$ and part of the case $n = 3$ are shown in the figure. It follows from the inductive definition that the vertices of simplices in the barycentric subdivision of $[v_0, \cdots, v_n]$ are exactly the barycenters of all the k-dimensional faces $[v_{i_0}, \cdots, v_{i_k}]$ of $[v_0, \cdots, v_n]$ for $0 \le k \le n$. When $k = 0$ this gives the original vertices v_i since the barycenter of a 0-simplex is itself. The barycenter of $[v_{i_0}, \cdots, v_{i_k}]$ has barycentric coordinates $t_i = 1/(k+1)$ for $i = i_0, \cdots, i_k$ and $t_i = 0$ otherwise.

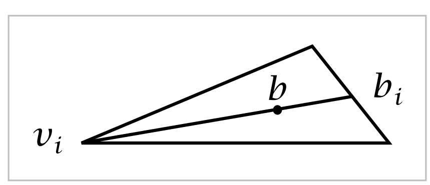


The n-simplices of the barycentric subdivision of Δ^n, together with all their faces, do in fact form a Δ-complex structure on Δ^n, indeed a simplicial complex structure, though we shall not need to know this in what follows.

A fact we will need is that the diameter of each simplex of the barycentric subdivision of $[v_0, \cdots, v_n]$ is at most $n/(n+1)$ times the diameter of $[v_0, \cdots, v_n]$. Here the diameter of a simplex is by definition the maximum distance between any two of its points, and we are using the metric from the ambient Euclidean space $\mathbb{R}^m$ containing $[v_0, \cdots, v_n]$. The diameter of a simplex equals the maximum distance between any of its vertices because the distance between two points v and $\sum_i t_i v_i$ of $[v_0, \cdots, v_n]$ satisfies the inequality

$$\left|v - \textstyle\sum_i t_i v_i\right| = \left|\textstyle\sum_i t_i (v - v_i)\right| \le \textstyle\sum_i t_i |v - v_i| \le \textstyle\sum_i t_i \max_j |v - v_j| = \max_j |v - v_j|$$

To obtain the bound $n/(n+1)$ on the ratio of diameters, we therefore need to verify that the distance between any two vertices w_j and w_k of a simplex $[w_0, \cdots, w_n]$ of the barycentric subdivision of $[v_0, \cdots, v_n]$ is at most $n/(n+1)$ times the diameter of $[v_0, \cdots, v_n]$. If neither w_j nor w_k is the barycenter b of $[v_0, \cdots, v_n]$, then these two points lie in a proper face of $[v_0, \cdots, v_n]$ and we are done by induction on n. So we may suppose w_j, say, is the barycenter b, and then by the previous displayed inequality we may take w_k to be a vertex v_i. Let b_i be the barycenter of $[v_0, \cdots, \hat{v}_i, \cdots, v_n]$, with all barycentric coordinates equal to $1/n$ except for $t_i = 0$. Then we have $b = \frac{1}{n+1} v_i + \frac{n}{n+1} b_i$. The sum of the two coefficients is 1, so b lies on the line segment $[v_i, b_i]$ from v_i to b_i, and the distance from b to v_i is $n/(n+1)$ times the length of $[v_i, b_i]$. Hence the distance from b to v_i is bounded by $n/(n+1)$ times the diameter of $[v_0, \cdots, v_n]$.



The significance of the factor $n/(n+1)$ is that by repeated barycentric subdivision we can produce simplices of arbitrarily small diameter since $(n/(n+1))^r$ approaches

0 as r goes to infinity. It is important that the bound $n/(n+1)$ does not depend on the shape of the simplex since repeated barycentric subdivision produces simplices of many different shapes.

(2) *Barycentric Subdivision of Linear Chains.* The main part of the proof will be to construct a subdivision operator $S : C_n(X) \to C_n(X)$ and show this is chain homotopic to the identity map. First we will construct S and the chain homotopy in a more restricted linear setting.

For a convex set Y in some Euclidean space, the linear maps $\Delta^n \to Y$ generate a subgroup of $C_n(Y)$ that we denote $LC_n(Y)$, the *linear chains*. The boundary map $\partial : C_n(Y) \to C_{n-1}(Y)$ takes $LC_n(Y)$ to $LC_{n-1}(Y)$, so the linear chains form a subcomplex of the singular chain complex of Y. We can uniquely designate a linear map $\lambda : \Delta^n \to Y$ by $[w_0, \cdots, w_n]$ where w_i is the image under λ of the i^{th} vertex of Δ^n. To avoid having to make exceptions for 0-simplices it will be convenient to augment the complex $LC(Y)$ by setting $LC_{-1}(Y) = \mathbb{Z}$ generated by the empty simplex $[\varnothing]$, with $\partial[w_0] = [\varnothing]$ for all 0-simplices $[w_0]$.

Each point $b \in Y$ determines a homomorphism $b : LC_n(Y) \to LC_{n+1}(Y)$ defined on basis elements by $b([w_0, \cdots, w_n]) = [b, w_0, \cdots, w_n]$. Geometrically, the homomorphism b can be regarded as a cone operator, sending a linear chain to the cone having the linear chain as the base of the cone and the point b as the tip of the cone. Applying the usual formula for ∂, we obtain the relation $\partial b([w_0, \cdots, w_n]) = [w_0, \cdots, w_n] - b(\partial[w_0, \cdots, w_n])$. By linearity it follows that $\partial b(\alpha) = \alpha - b(\partial\alpha)$ for all $\alpha \in LC_n(Y)$. This expresses algebraically the geometric fact that the boundary of a cone consists of its base together with the cone on the boundary of its base. The relation $\partial b(\alpha) = \alpha - b(\partial\alpha)$ can be rewritten as $\partial b + b\partial = \mathbb{1}$, so b is a chain homotopy between the identity map and the zero map on the augmented chain complex $LC(Y)$.

Now we define a subdivision homomorphism $S : LC_n(Y) \to LC_n(Y)$ by induction on n. Let $\lambda : \Delta^n \to Y$ be a generator of $LC_n(Y)$ and let b_λ be the image of the barycenter of Δ^n under λ. Then the inductive formula for S is $S(\lambda) = b_\lambda(S\partial\lambda)$ where $b_\lambda : LC_{n-1}(Y) \to LC_n(Y)$ is the cone operator defined in the preceding paragraph. The induction starts with $S([\varnothing]) = [\varnothing]$, so S is the identity on $LC_{-1}(Y)$. It is also the identity on $LC_0(Y)$, since when $n = 0$ the formula for S becomes $S([w_0]) = w_0(S\partial[w_0]) = w_0(S([\varnothing])) = w_0([\varnothing]) = [w_0]$. When λ is an embedding, with image a genuine n-simplex $[w_0, \cdots, w_n]$, then $S(\lambda)$ is the sum of the n-simplices in the barycentric subdivision of $[w_0, \cdots, w_n]$, with certain signs that could be computed explicitly. This is apparent by comparing the inductive definition of S with the inductive definition of the barycentric subdivision of a simplex.

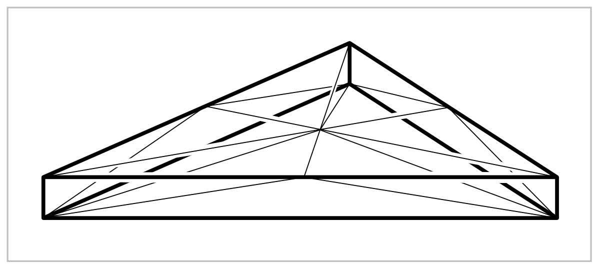
Let us check that the maps S satisfy $\partial S = S\partial$, and hence give a chain map from the chain complex $LC(Y)$ to itself. Since $S = \mathbb{1}$ on $LC_0(Y)$ and $LC_{-1}(Y)$, we certainly have $\partial S = S\partial$ on $LC_0(Y)$. The result for larger n is given by the following calculation, in which we omit some parentheses to unclutter the formulas:

$$\begin{aligned}\partial S\lambda &= \partial b_\lambda(S\partial\lambda)\\ &= S\partial\lambda - b_\lambda\partial(S\partial\lambda) \qquad \text{since } \partial b_\lambda = 1\!\!1 - b_\lambda\partial\\ &= S\partial\lambda - b_\lambda S(\partial\partial\lambda) \qquad \text{since } \partial S(\partial\lambda) = S\partial(\partial\lambda) \text{ by induction on } n\\ &= S\partial\lambda \qquad \text{since } \partial\partial = 0\end{aligned}$$

We next build a chain homotopy $T:LC_n(Y)\to LC_{n+1}(Y)$ between S and the identity, fitting into a diagram

$$\begin{tikzcd}
\cdots \arrow[r] & LC_2(Y) \arrow[r] \arrow[d, "S"] & LC_1(Y) \arrow[r] \arrow[d, "S"] \arrow[dl, "T"'] & LC_0(Y) \arrow[r] \arrow[d, "S"', "1\!\!1"] \arrow[dl, "T"'] & LC_{-1}(Y) \arrow[r] \arrow[d, "S"', "1\!\!1"] \arrow[dl, "T"', "0"] & 0 \\
\cdots \arrow[r] & LC_2(Y) \arrow[r] & LC_1(Y) \arrow[r] & LC_0(Y) \arrow[r] & LC_{-1}(Y) \arrow[r] & 0
\end{tikzcd}$$

We define T on $LC_n(Y)$ inductively by setting $T=0$ for $n=-1$ and letting $T\lambda = b_\lambda(\lambda - T\partial\lambda)$ for $n\ge 0$. The geometric motivation for this formula is an inductively defined subdivision of $\Delta^n\times I$ obtained by joining all simplices in $\Delta^n\times\{0\}\cup\partial\Delta^n\times I$ to the barycenter of $\Delta^n\times\{1\}$, as indicated in the figure in the case $n=2$. What T actually does is take the image of this subdivision under the projection $\Delta^n\times I\to\Delta^n$.



The chain homotopy formula $\partial T + T\partial = 1\!\!1 - S$ is trivial on $LC_{-1}(Y)$ where $T=0$ and $S = 1\!\!1$. Verifying the formula on $LC_n(Y)$ with $n\ge 0$ is done by the calculation

$$\begin{aligned}\partial T\lambda &= \partial b_\lambda(\lambda - T\partial\lambda)\\ &= \lambda - T\partial\lambda - b_\lambda\partial(\lambda - T\partial\lambda) \qquad \text{since } \partial b_\lambda = 1\!\!1 - b_\lambda\partial\\ &= \lambda - T\partial\lambda - b_\lambda[\partial\lambda - \partial T(\partial\lambda)]\\ &= \lambda - T\partial\lambda - b_\lambda[S(\partial\lambda) + T\partial(\partial\lambda)] \qquad \text{by induction on } n\\ &= \lambda - T\partial\lambda - S\lambda \qquad \text{since } \partial\partial = 0 \text{ and } S\lambda = b_\lambda(S\partial\lambda)\end{aligned}$$

Now we can discard the group $LC_{-1}(Y)$ and the relation $\partial T + T\partial = 1\!\!1 - S$ still holds since T was zero on $LC_{-1}(Y)$.

(3) *Barycentric Subdivision of General Chains.* Define $S:C_n(X)\to C_n(X)$ by setting $S\sigma = \sigma_\sharp S\Delta^n$ for a singular n-simplex $\sigma:\Delta^n\to X$. Since $S\Delta^n$ is the sum of the n-simplices in the barycentric subdivision of Δ^n, with certain signs, $S\sigma$ is the corresponding signed sum of the restrictions of σ to the n-simplices of the barycentric subdivision of Δ^n. The operator S is a chain map since

$$\begin{aligned}\partial S\sigma &= \partial\sigma_\sharp S\Delta^n = \sigma_\sharp\partial S\Delta^n = \sigma_\sharp S\partial\Delta^n\\ &= \sigma_\sharp S\big(\textstyle\sum_i(-1)^i\Delta_i^n\big) \qquad \text{where } \Delta_i^n \text{ is the } i^{th} \text{ face of } \Delta^n\\ &= \textstyle\sum_i(-1)^i\sigma_\sharp S\Delta_i^n\\ &= \textstyle\sum_i(-1)^i S(\sigma|\Delta_i^n)\\ &= S\big(\textstyle\sum_i(-1)^i\sigma|\Delta_i^n\big) = S(\partial\sigma)\end{aligned}$$

In similar fashion we define $T: C_n(X) \to C_{n+1}(X)$ by $T\sigma = \sigma_\sharp T\Delta^n$, and this gives a chain homotopy between S and the identity, since the formula $\partial T + T\partial = 1\!\!1 - S$ holds by the calculation

$$\begin{aligned}\partial T\sigma = \partial\sigma_\sharp T\Delta^n = \sigma_\sharp \partial T\Delta^n = \sigma_\sharp(\Delta^n - S\Delta^n - T\partial\Delta^n) &= \sigma - S\sigma - \sigma_\sharp T\partial\Delta^n \\ &= \sigma - S\sigma - T(\partial\sigma)\end{aligned}$$

where the last equality follows just as in the previous displayed calculation, with S replaced by T.

(4) *Iterated Barycentric Subdivision.* A chain homotopy between $1\!\!1$ and the iterate S^m is given by the operator $D_m = \sum_{0 \le i < m} TS^i$ since

$$\partial D_m + D_m\partial = \sum_{0\le i<m} (\partial TS^i + TS^i\partial) = \sum_{0\le i<m} (\partial TS^i + T\partial S^i) =$$
$$\sum_{0\le i<m} (\partial T + T\partial)S^i = \sum_{0\le i<m} (1\!\!1 - S)S^i = \sum_{0\le i<m} (S^i - S^{i+1}) = 1\!\!1 - S^m$$

For each singular n-simplex $\sigma: \Delta^n \to X$ there exists an m such that $S^m(\sigma)$ lies in $C_n^{\mathcal{U}}(X)$ since the diameter of the simplices of $S^m(\Delta^n)$ will be less than a Lebesgue number of the cover of Δ^n by the open sets $\sigma^{-1}(\operatorname{int} U_j)$ if m is large enough. (Recall that a Lebesgue number for an open cover of a compact metric space is a number $\varepsilon > 0$ such that every set of diameter less than ε lies in some set of the cover; such a number exists by an elementary compactness argument.) We cannot expect the same number m to work for all σ's, so let us define $m(\sigma)$ to be the smallest m such that $S^m\sigma$ is in $C_n^{\mathcal{U}}(X)$.

We now define $D: C_n(X) \to C_{n+1}(X)$ by setting $D\sigma = D_{m(\sigma)}\sigma$ for each singular n-simplex $\sigma: \Delta^n \to X$. For this D we would like to find a chain map $\rho: C_n(X) \to C_n(X)$ with image in $C_n^{\mathcal{U}}(X)$ satisfying the chain homotopy equation

$$(*) \qquad \partial D + D\partial = 1\!\!1 - \rho$$

A quick way to do this is simply to regard this equation as defining ρ, so we let $\rho = 1\!\!1 - \partial D - D\partial$. It follows easily that ρ is a chain map since

$$\partial\rho(\sigma) = \partial\sigma - \partial^2 D\sigma - \partial D\partial\sigma = \partial\sigma - \partial D\partial\sigma$$
$$\text{and} \qquad \rho(\partial\sigma) = \partial\sigma - \partial D\partial\sigma - D\partial^2\sigma = \partial\sigma - \partial D\partial\sigma$$

To check that ρ takes $C_n(X)$ to $C_n^{\mathcal{U}}(X)$ we compute $\rho(\sigma)$ more explicitly:

$$\begin{aligned}\rho(\sigma) &= \sigma - \partial D\sigma - D(\partial\sigma) \\ &= \sigma - \partial D_{m(\sigma)}\sigma - D(\partial\sigma) \\ &= S^{m(\sigma)}\sigma + D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma) \qquad \text{since} \quad \partial D_m + D_m\partial = 1\!\!1 - S^m\end{aligned}$$

The term $S^{m(\sigma)}\sigma$ lies in $C_n^{\mathcal{U}}(X)$ by the definition of $m(\sigma)$. The remaining terms $D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma)$ are linear combinations of terms $D_{m(\sigma)}(\sigma_j) - D_{m(\sigma_j)}(\sigma_j)$ for σ_j the restriction of σ to a face of Δ^n, so $m(\sigma_j) \le m(\sigma)$ and hence the difference

$D_{m(\sigma)}(\sigma_j) - D_{m(\sigma_j)}(\sigma_j)$ consists of terms $TS^i(\sigma_j)$ with $i \geq m(\sigma_j)$, and these terms lie in $C_n^{\mathcal{U}}(X)$ since T takes $C_{n-1}^{\mathcal{U}}(X)$ to $C_n^{\mathcal{U}}(X)$.

Viewing ρ as a chain map $C_n(X) \to C_n^{\mathcal{U}}(X)$, the equation $(*)$ says that $\partial D + D\partial = \mathbb{1} - \iota\rho$ for $\iota : C_n^{\mathcal{U}}(X) \hookrightarrow C_n(X)$ the inclusion. Furthermore, $\rho\iota = \mathbb{1}$ since D is identically zero on $C_n^{\mathcal{U}}(X)$, as $m(\sigma) = 0$ if σ is in $C_n^{\mathcal{U}}(X)$, hence the summation defining $D\sigma$ is empty. Thus we have shown that ρ is a chain homotopy inverse for ι. $\square$

Proof of the Excision Theorem: We prove the second version, involving a decomposition $X = A \cup B$. For the cover $\mathcal{U} = \{A, B\}$ we introduce the suggestive notation $C_n(A+B)$ for $C_n^{\mathcal{U}}(X)$, the sums of chains in A and chains in B. At the end of the preceding proof we had formulas $\partial D + D\partial = \mathbb{1} - \iota\rho$ and $\rho\iota = \mathbb{1}$. All the maps appearing in these formulas take chains in A to chains in A, so they induce quotient maps when we factor out chains in A. These quotient maps automatically satisfy the same two formulas, so the inclusion $C_n(A+B)/C_n(A) \hookrightarrow C_n(X)/C_n(A)$ induces an isomorphism on homology. The map $C_n(B)/C_n(A \cap B) \to C_n(A+B)/C_n(A)$ induced by inclusion is obviously an isomorphism since both quotient groups are free with basis the singular n-simplices in B that do not lie in A. Hence we obtain the desired isomorphism $H_n(B, A \cap B) \approx H_n(X, A)$ induced by inclusion. $\square$

All that remains in the proof of Theorem 2.13 is to replace relative homology groups with absolute homology groups. This is achieved by the following result.

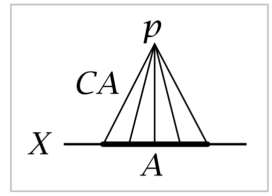
Proposition 2.22. *For good pairs (X, A), the quotient map $q : (X, A) \to (X/A, A/A)$ induces isomorphisms $q_* : H_n(X, A) \to H_n(X/A, A/A) \approx \widetilde{H}_n(X/A)$ for all n.*

Proof: Let V be a neighborhood of A in X that deformation retracts onto A. We have a commutative diagram

$$\begin{array}{ccccc}
H_n(X,A) & \longrightarrow & H_n(X,V) & \longleftarrow & H_n(X-A,V-A) \\
\Big\downarrow q_* & & \Big\downarrow q_* & & \Big\downarrow q_* \\
H_n(X/A,A/A) & \longrightarrow & H_n(X/A,V/A) & \longleftarrow & H_n(X/A-A/A,V/A-A/A)
\end{array}$$

The upper left horizontal map is an isomorphism since in the long exact sequence of the triple (X, V, A) the groups $H_n(V, A)$ are zero for all n, because a deformation retraction of V onto A gives a homotopy equivalence of pairs $(V, A) \simeq (A, A)$, and $H_n(A, A) = 0$. The deformation retraction of V onto A induces a deformation retraction of V/A onto A/A, so the same argument shows that the lower left horizontal map is an isomorphism as well. The other two horizontal maps are isomorphisms directly from excision. The right-hand vertical map q_* is an isomorphism since q restricts to a homeomorphism on the complement of A. From the commutativity of the diagram it follows that the left-hand q_* is an isomorphism. $\square$

This proposition shows that relative homology can be expressed as reduced absolute homology in the case of good pairs (X, A), but in fact there is a way of doing this for arbitrary pairs. Consider the space $X \cup CA$ where CA is the cone $(A\times I)/(A\times\{0\})$ whose base $A\times\{1\}$ we identify with $A \subset X$. Using terminology introduced in Chapter 0, $X\cup CA$ can also be described as the mapping cone of the inclusion $A \hookrightarrow X$. The assertion is that $H_n(X,A)$ is isomorphic to $\widetilde{H}_n(X \cup CA)$ for all n via the sequence of isomorphisms



$$\widetilde{H}_n(X \cup CA) \approx H_n(X \cup CA, CA) \approx H_n(X \cup CA - \{p\}, CA - \{p\}) \approx H_n(X,A)$$

where $p \in CA$ is the tip of the cone. The first isomorphism comes from the exact sequence of the pair, using the fact that CA is contractible. The second isomorphism is excision, and the third comes from a deformation retraction of $CA - \{p\}$ onto A.

Here is an application of the preceding proposition:

Example 2.23. Let us find explicit cycles representing generators of the infinite cyclic groups $H_n(D^n,\partial D^n)$ and $\widetilde{H}_n(S^n)$. Replacing $(D^n,\partial D^n)$ by the equivalent pair $(\Delta^n,\partial\Delta^n)$, we will show by induction on n that the identity map $i_n:\Delta^n\to\Delta^n$, viewed as a singular n-simplex, is a cycle generating $H_n(\Delta^n,\partial\Delta^n)$. That it is a cycle is clear since we are considering relative homology. When $n = 0$ it certainly represents a generator. For the induction step, let $\Lambda \subset \Delta^n$ be the union of all but one of the $(n-1)$-dimensional faces of Δ^n. Then we claim there are isomorphisms

$$H_n(\Delta^n,\partial\Delta^n) \xrightarrow{\approx} H_{n-1}(\partial\Delta^n,\Lambda) \xleftarrow{\approx} H_{n-1}(\Delta^{n-1},\partial\Delta^{n-1})$$

The first isomorphism is a boundary map in the long exact sequence of the triple $(\Delta^n,\partial\Delta^n,\Lambda)$, whose third terms $H_i(\Delta^n,\Lambda)$ are zero since Δ^n deformation retracts onto Λ, hence $(\Delta^n,\Lambda) \simeq (\Lambda,\Lambda)$. The second isomorphism is induced by the inclusion $i:\Delta^{n-1}\to\partial\Delta^n$ as the face not contained in Λ. When $n = 1$, i induces an isomorphism on relative homology since this is true already at the chain level. When $n > 1$, $\partial\Delta^{n-1}$ is nonempty so we are dealing with good pairs and i induces a homeomorphism of quotients $\Delta^{n-1}/\partial\Delta^{n-1} \approx \partial\Delta^n/\Lambda$. The induction step then follows since the cycle i_n is sent under the first isomorphism to the cycle ∂i_n which equals $\pm i_{n-1}$ in $C_{n-1}(\partial\Delta^n,\Lambda)$.

To find a cycle generating $\widetilde{H}_n(S^n)$ let us regard S^n as two n-simplices Δ_1^n and Δ_2^n with their boundaries identified in the obvious way, preserving the ordering of vertices. The difference $\Delta_1^n - \Delta_2^n$, viewed as a singular n-chain, is then a cycle, and we claim it represents a generator of $\widetilde{H}_n(S^n)$. To see this, consider the isomorphisms

$$\widetilde{H}_n(S^n) \xrightarrow{\approx} H_n(S^n,\Delta_2^n) \xleftarrow{\approx} H_n(\Delta_1^n,\partial\Delta_1^n)$$

where the first isomorphism comes from the long exact sequence of the pair (S^n,Δ_2^n) and the second isomorphism is justified in the nontrivial cases $n > 0$ by passing to quotients as before. Under these isomorphisms the cycle $\Delta_1^n - \Delta_2^n$ in the first group corresponds to the cycle Δ_1^n in the third group, which represents a generator of this group as we have seen, so $\Delta_1^n - \Delta_2^n$ represents a generator of $\widetilde{H}_n(S^n)$.

The preceding proposition implies that the excision property holds also for subcomplexes of CW complexes:

Corollary 2.24. *If the CW complex X is the union of subcomplexes A and B, then the inclusion $(B, A\cap B) \hookrightarrow (X, A)$ induces isomorphisms $H_n(B, A\cap B) \to H_n(X, A)$ for all n.*

Proof: Since CW pairs are good, Proposition 2.22 allows us to pass to the quotient spaces $B/(A\cap B)$ and X/A which are homeomorphic, assuming we are not in the trivial case $A\cap B = \varnothing$. $\square$

Here is another application of the preceding proposition:

Corollary 2.25. *For a wedge sum $\bigvee_\alpha X_\alpha$, the inclusions $i_\alpha : X_\alpha \hookrightarrow \bigvee_\alpha X_\alpha$ induce an isomorphism $\bigoplus_\alpha i_{\alpha*} : \bigoplus_\alpha \widetilde{H}_n(X_\alpha) \to \widetilde{H}_n(\bigvee_\alpha X_\alpha)$, provided that the wedge sum is formed at basepoints $x_\alpha \in X_\alpha$ such that the pairs (X_α, x_α) are good.*

Proof: Since reduced homology is the same as homology relative to a basepoint, this follows from the proposition by taking $(X, A) = (\coprod_\alpha X_\alpha, \coprod_\alpha \{x_\alpha\})$. $\square$

Here is an application of the machinery we have developed, a classical result of Brouwer from around 1910 known as 'invariance of dimension', which says in particular that $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^n$ if $m \neq n$.

Theorem 2.26. *If nonempty open sets $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are homeomorphic, then $m = n$.*

Proof: For $x \in U$ we have $H_k(U, U - \{x\}) \approx H_k(\mathbb{R}^m, \mathbb{R}^m - \{x\})$ by excision. From the long exact sequence for the pair $(\mathbb{R}^m, \mathbb{R}^m - \{x\})$ we get $H_k(\mathbb{R}^m, \mathbb{R}^m - \{x\}) \approx \widetilde{H}_{k-1}(\mathbb{R}^m - \{x\})$. Since $\mathbb{R}^m - \{x\}$ deformation retracts onto a sphere S^{m-1}, we conclude that $H_k(U, U - \{x\})$ is $\mathbb{Z}$ for $k = m$ and 0 otherwise. By the same reasoning, $H_k(V, V - \{y\})$ is $\mathbb{Z}$ for $k = n$ and 0 otherwise. Since a homeomorphism $h : U \to V$ induces isomorphisms $H_k(U, U - \{x\}) \to H_k(V, V - \{h(x)\})$ for all k, we must have $m = n$. $\square$

Generalizing the idea of this proof, the **local homology groups** of a space X at a point $x \in X$ are defined to be the groups $H_n(X, X - \{x\})$. For any open neighborhood U of x, excision gives isomorphisms $H_n(X, X - \{x\}) \approx H_n(U, U - \{x\})$ assuming points are closed in X, and thus the groups $H_n(X, X - \{x\})$ depend only on the local topology of X near x. A homeomorphism $f : X \to Y$ must induce isomorphisms $H_n(X, X - \{x\}) \approx H_n(Y, Y - \{f(x)\})$ for all x and n, so the local homology groups can be used to tell when spaces are not locally homeomorphic at certain points, as in the preceding proof. The exercises give some further examples of this.

Naturality

The exact sequences we have been constructing have an extra property that will become important later at key points in many arguments, though at first glance this property may seem just an idle technicality, not very interesting. We shall discuss the property now rather than interrupting later arguments to check it when it is needed, but the reader may prefer to postpone a careful reading of this discussion.

The property is called **naturality**. For example, to say that the long exact sequence of a pair is natural means that for a map $f:(X,A)\to(Y,B)$, the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r, "i_*"] \arrow[d, "f_*"] & H_n(X) \arrow[r, "j_*"] \arrow[d, "f_*"] & H_n(X,A) \arrow[r, "\partial"] \arrow[d, "f_*"] & H_{n-1}(A) \arrow[r] \arrow[d, "f_*"] & \cdots \\
\cdots \arrow[r] & H_n(B) \arrow[r, "i_*"] & H_n(Y) \arrow[r, "j_*"] & H_n(Y,B) \arrow[r, "\partial"] & H_{n-1}(B) \arrow[r] & \cdots
\end{tikzcd}$$

is commutative. Commutativity of the squares involving i_* and j_* follows from the obvious commutativity of the corresponding squares of chain groups, with C_n in place of H_n. For the other square, when we defined induced homomorphisms we saw that $f_\sharp\partial = \partial f_\sharp$ at the chain level. Then for a class $[\alpha]\in H_n(X,A)$ represented by a relative cycle α, we have $f_*\partial[\alpha] = f_*[\partial\alpha] = [f_\sharp\partial\alpha] = [\partial f_\sharp\alpha] = \partial[f_\sharp\alpha] = \partial f_*[\alpha]$.

Alternatively, we could appeal to the general algebraic fact that the long exact sequence of homology groups associated to a short exact sequence of chain complexes is natural: For a commutative diagram of short exact sequences of chain complexes

$$\begin{tikzcd}[row sep=small, column sep=small]
 & & 0 \arrow[dd] & & 0 \arrow[dd] & & 0 \arrow[dd] & \\
 & 0 \arrow[dd] & & 0 \arrow[dd] & & 0 \arrow[dd] & & \\
\cdots \arrow[rr] & & A'_{n+1} \arrow[rr, "\partial"] \arrow[dd, "i'"] & & A'_n \arrow[rr, "\partial"] \arrow[dd, "i'"] & & A'_{n-1} \arrow[r] \arrow[dd, "i'"] & \cdots \\
\cdots \arrow[r] & A_{n+1} \arrow[ur, "\alpha"] \arrow[rr, "\partial"] \arrow[dd, "i"] & & A_n \arrow[ur, "\alpha"] \arrow[rr, "\partial"] \arrow[dd, "i"] & & A_{n-1} \arrow[ur, "\alpha"] \arrow[rr] \arrow[dd, "i"] & & \cdots \\
\cdots \arrow[rr] & & B'_{n+1} \arrow[rr, "\partial"] \arrow[dd, "j'"] & & B'_n \arrow[rr, "\partial"] \arrow[dd, "j'"] & & B'_{n-1} \arrow[r] \arrow[dd, "j'"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[ur, "\beta"] \arrow[rr, "\partial"] \arrow[dd, "j"] & & B_n \arrow[ur, "\beta"] \arrow[rr, "\partial"] \arrow[dd, "j"] & & B_{n-1} \arrow[ur, "\beta"] \arrow[rr] \arrow[dd, "j"] & & \cdots \\
\cdots \arrow[rr] & & C'_{n+1} \arrow[rr, "\partial"] \arrow[dd] & & C'_n \arrow[rr, "\partial"] \arrow[dd] & & C'_{n-1} \arrow[r] \arrow[dd] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[ur, "\gamma"] \arrow[rr, "\partial"] \arrow[dd] & & C_n \arrow[ur, "\gamma"] \arrow[rr, "\partial"] \arrow[dd] & & C_{n-1} \arrow[ur, "\gamma"] \arrow[rr] \arrow[dd] & & \cdots \\
 & & 0 & & 0 & & 0 & \\
 & 0 & & 0 & & 0 & &
\end{tikzcd}$$

the induced diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r, "i_*"] \arrow[d, "\alpha_*"] & H_n(B) \arrow[r, "j_*"] \arrow[d, "\beta_*"] & H_n(C) \arrow[r, "\partial"] \arrow[d, "\gamma_*"] & H_{n-1}(A) \arrow[r] \arrow[d, "\alpha_*"] & \cdots \\
\cdots \arrow[r] & H_n(A') \arrow[r, "i'_*"] & H_n(B') \arrow[r, "j'_*"] & H_n(C') \arrow[r, "\partial"] & H_{n-1}(A') \arrow[r] & \cdots
\end{tikzcd}$$

is commutative. Commutativity of the first two squares is obvious since $\beta i = i'\alpha$ implies $\beta_* i_* = i'_*\alpha_*$ and $\gamma j = j'\beta$ implies $\gamma_* j_* = j'_*\beta_*$. For the third square, recall that the map $\partial: H_n(C)\to H_{n-1}(A)$ was defined by $\partial[c] = [a]$ where $c = j(b)$ and $i(a) = \partial b$. Then $\partial[\gamma(c)] = [\alpha(a)]$ since $\gamma(c) = \gamma j(b) = j'(\beta(b))$ and $i'(\alpha(a)) = \beta i(a) = \beta\partial(b) = \partial\beta(b)$. Hence $\partial\gamma_*[c] = \alpha_*[a] = \alpha_*\partial[c]$.

This algebraic fact also implies naturality of the long exact sequence of a triple and the long exact sequence of reduced homology of a pair.

Finally, there is the naturality of the long exact sequence in Theorem 2.13, that is, commutativity of the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \widetilde{H}_n(A) \arrow[r, "i_*"] \arrow[d, "f_*"] & \widetilde{H}_n(X) \arrow[r, "q_*"] \arrow[d, "f_*"] & \widetilde{H}_n(X/A) \arrow[r, "\partial"] \arrow[d, "\overline{f}_*"] & \widetilde{H}_{n-1}(A) \arrow[r] \arrow[d, "f_*"] & \cdots \\
\cdots \arrow[r] & \widetilde{H}_n(B) \arrow[r, "i_*"] & \widetilde{H}_n(Y) \arrow[r, "q_*"] & \widetilde{H}_n(Y/B) \arrow[r, "\partial"] & \widetilde{H}_{n-1}(B) \arrow[r] & \cdots
\end{tikzcd}$$

where i and q denote inclusions and quotient maps, and $\overline{f}: X/A \to Y/B$ is induced by f. The first two squares commute since $fi = if$ and $\overline{f}q = qf$. The third square expands into

$$\begin{tikzcd}
\widetilde{H}_n(X/A) \arrow[r, "j_*", "\approx"'] \arrow[d, "\overline{f}_*"] & H_n(X/A, A/A) \arrow[d, "\overline{f}_*"] & H_n(X,A) \arrow[l, "q_*"', "\approx"] \arrow[r, "\partial"] \arrow[d, "f_*"] & \widetilde{H}_{n-1}(A) \arrow[d, "f_*"] \\
\widetilde{H}_n(Y/B) \arrow[r, "j_*", "\approx"'] & H_n(Y/B, B/B) & H_n(Y,B) \arrow[l, "q_*"', "\approx"] \arrow[r, "\partial"] & \widetilde{H}_{n-1}(B)
\end{tikzcd}$$

We have already shown commutativity of the first and third squares, and the second square commutes since $\overline{f}q = qf$.

The Equivalence of Simplicial and Singular Homology

We can use the preceding results to show that the simplicial and singular homology groups of Δ-complexes are always isomorphic. For the proof it will be convenient to consider the relative case as well, so let X be a Δ-complex with $A \subset X$ a subcomplex. Thus A is the Δ-complex formed by any union of simplices of X. Relative groups $H_n^\Delta(X,A)$ can be defined in the same way as for singular homology, via relative chains $\Delta_n(X,A) = \Delta_n(X)/\Delta_n(A)$, and this yields a long exact sequence of simplicial homology groups for the pair (X,A) by the same algebraic argument as for singular homology. There is a canonical homomorphism $H_n^\Delta(X,A) \to H_n(X,A)$ induced by the chain map $\Delta_n(X,A) \to C_n(X,A)$ sending each n-simplex of X to its characteristic map $\sigma: \Delta^n \to X$. The possibility $A = \varnothing$ is not excluded, in which case the relative groups reduce to absolute groups.

Theorem 2.27. *The homomorphisms $H_n^\Delta(X,A) \to H_n(X,A)$ are isomorphisms for all n and all Δ-complex pairs (X,A).*

Proof: First we do the case that X is finite-dimensional and A is empty. For X^k the k-skeleton of X, consisting of all simplices of dimension k or less, we have a commutative diagram of exact sequences:

$$\begin{tikzcd}
H_{n+1}^\Delta(X^k, X^{k-1}) \arrow[r] \arrow[d] & H_n^\Delta(X^{k-1}) \arrow[r] \arrow[d] & H_n^\Delta(X^k) \arrow[r] \arrow[d] & H_n^\Delta(X^k, X^{k-1}) \arrow[r] \arrow[d] & H_{n-1}^\Delta(X^{k-1}) \arrow[d] \\
H_{n+1}(X^k, X^{k-1}) \arrow[r] & H_n(X^{k-1}) \arrow[r] & H_n(X^k) \arrow[r] & H_n(X^k, X^{k-1}) \arrow[r] & H_{n-1}(X^{k-1})
\end{tikzcd}$$

Let us first show that the first and fourth vertical maps are isomorphisms for all n. The simplicial chain group $\Delta_n(X^k,X^{k-1})$ is zero for $n \neq k$, and is free abelian with basis the k-simplices of X when $n = k$. Hence $H_n^{\Delta}(X^k,X^{k-1})$ has exactly the same description. The corresponding singular homology groups $H_n(X^k,X^{k-1})$ can be computed by considering the map $\Phi:\coprod_\alpha(\Delta^k_\alpha,\partial\Delta^k_\alpha)\to(X^k,X^{k-1})$ formed by the characteristic maps $\Delta^k\to X$ for all the k-simplices of X. Since Φ induces a homeomorphism of quotient spaces $\coprod_\alpha\Delta^k_\alpha/\coprod_\alpha\partial\Delta^k_\alpha \approx X^k/X^{k-1}$, it induces isomorphisms on all singular homology groups. Thus $H_n(X^k,X^{k-1})$ is zero for $n \neq k$, while for $n = k$ this group is free abelian with basis represented by the relative cycles given by the characteristic maps of all the k-simplices of X, in view of the fact that $H_k(\Delta^k,\partial\Delta^k)$ is generated by the identity map $\Delta^k\to\Delta^k$, as we showed in Example 2.23. Therefore the map $H_k^{\Delta}(X^k,X^{k-1})\to H_k(X^k,X^{k-1})$ is an isomorphism.

By induction on k we may assume the second and fifth vertical maps in the preceding diagram are isomorphisms as well. The following frequently quoted basic algebraic lemma will then imply that the middle vertical map is an isomorphism, finishing the proof when X is finite-dimensional and $A = \varnothing$.

The Five-Lemma. *In a commutative diagram of abelian groups as at the right, if the two rows are exact and α, β, δ, and ε are isomorphisms, then γ is an isomorphism also.*

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "\alpha"] & B \arrow[r, "j"] \arrow[d, "\beta"] & C \arrow[r, "k"] \arrow[d, "\gamma"] & D \arrow[r, "\ell"] \arrow[d, "\delta"] & E \arrow[d, "\varepsilon"] \\
A' \arrow[r, "i'"] & B' \arrow[r, "j'"] & C' \arrow[r, "k'"] & D' \arrow[r, "\ell'"] & E'
\end{tikzcd}$$

Proof: It suffices to show:

(a) γ is surjective if β and δ are surjective and ε is injective.

(b) γ is injective if β and δ are injective and α is surjective.

The proofs of these two statements are straightforward diagram chasing. There is really no choice about how the argument can proceed, and it would be a good exercise for the reader to close the book now and reconstruct the proofs without looking.

To prove (a), start with an element $c' \in C'$. Then $k'(c') = \delta(d)$ for some $d \in D$ since δ is surjective. Since ε is injective and $\varepsilon\ell(d) = \ell'\delta(d) = \ell'k'(c') = 0$, we deduce that $\ell(d) = 0$, hence $d = k(c)$ for some $c \in C$ by exactness of the upper row. The difference $c' - \gamma(c)$ maps to 0 under k' since $k'(c') - k'\gamma(c) = k'(c') - \delta k(c) = k'(c') - \delta(d) = 0$. Therefore $c' - \gamma(c) = j'(b')$ for some $b' \in B'$ by exactness. Since β is surjective, $b' = \beta(b)$ for some $b \in B$, and then $\gamma(c + j(b)) = \gamma(c) + \gamma j(b) = \gamma(c) + j'\beta(b) = \gamma(c) + j'(b') = c'$, showing that γ is surjective.

To prove (b), suppose that $\gamma(c) = 0$. Since δ is injective, $\delta k(c) = k'\gamma(c) = 0$ implies $k(c) = 0$, so $c = j(b)$ for some $b \in B$. The element $\beta(b)$ satisfies $j'\beta(b) = \gamma j(b) = \gamma(c) = 0$, so $\beta(b) = i'(a')$ for some $a' \in A'$. Since α is surjective, $a' = \alpha(a)$ for some $a \in A$. Since β is injective, $\beta(i(a) - b) = \beta i(a) - \beta(b) = i'\alpha(a) - \beta(b) = i'(a') - \beta(b) = 0$ implies $i(a) - b = 0$. Thus $b = i(a)$, and hence $c = j(b) = ji(a) = 0$ since $ji = 0$. This shows γ has trivial kernel. $\square$

Returning to the proof of the theorem, we next consider the case that X is infinite-dimensional, where we will use the following fact: A compact set in X can meet only finitely many open simplices of X, that is, simplices with their proper faces deleted. This is a general fact about CW complexes proved in the Appendix, but here is a direct proof for Δ-complexes. If a compact set C intersected infinitely many open simplices, it would contain an infinite sequence of points x_i each lying in a different open simplex. Then the sets $U_i = X - \bigcup_{j\neq i}\{x_j\}$, which are open since their preimages under the characteristic maps of all the simplices are clearly open, form an open cover of C with no finite subcover.

This can be applied to show the map $H_n^\Delta(X)\to H_n(X)$ is surjective. Represent a given element of $H_n(X)$ by a singular n-cycle z. This is a linear combination of finitely many singular simplices with compact images, meeting only finitely many open simplices of X, hence contained in X^k for some k. We have shown that $H_n^\Delta(X^k)\to H_n(X^k)$ is an isomorphism, in particular surjective, so z is homologous in X^k (hence in X) to a simplicial cycle. This gives surjectivity. Injectivity is similar: If a simplicial n-cycle z is the boundary of a singular chain in X, this chain has compact image and hence must lie in some X^k, so z represents an element of the kernel of $H_n^\Delta(X^k)\to H_n(X^k)$. But we know this map is injective, so z is a simplicial boundary in X^k, and therefore in X.

It remains to do the case of arbitrary X with $A \neq \varnothing$, but this follows from the absolute case by applying the five-lemma to the canonical map from the long exact sequence of simplicial homology groups for the pair (X,A) to the corresponding long exact sequence of singular homology groups. $\square$

We can deduce from this theorem that $H_n(X)$ is finitely generated whenever X is a Δ-complex with finitely many n-simplices, since in this case the simplicial chain group $\Delta_n(X)$ is finitely generated, hence also its subgroup of cycles and therefore also the latter group's quotient $H_n^\Delta(X)$. If we write $H_n(X)$ as the direct sum of cyclic groups, then the number of $\mathbb{Z}$ summands is known traditionally as the n^{th} **Betti number** of X, and integers specifying the orders of the finite cyclic summands are called **torsion coefficients**.

It is a curious historical fact that homology was not thought of originally as a sequence of groups, but rather as Betti numbers and torsion coefficients. One can after all compute Betti numbers and torsion coefficients from the simplicial boundary maps without actually mentioning homology groups. This computational viewpoint, with homology being numbers rather than groups, prevailed from when Poincaré first started serious work on homology around 1900, up until the 1920s when the more abstract viewpoint of groups entered the picture. During this period 'homology' meant primarily 'simplicial homology', and it was another 20 years before the shift to singular homology was complete, with the final definition of singular homology emerging only in a 1944 paper of Eilenberg, after contributions from quite a few others, particularly Alexander and Lefschetz. Within the next few years the rest of the basic structure of homology theory as we have presented it fell into place, and the first definitive treatment appeared in the classic book [Eilenberg & Steenrod 1952].

Exercises

1. What familiar space is the quotient Δ-complex of a 2-simplex $[v_0, v_1, v_2]$ obtained by identifying the edges $[v_0, v_1]$ and $[v_1, v_2]$, preserving the ordering of vertices?

2. Show that the Δ-complex obtained from Δ^3 by performing the order-preserving edge identifications $[v_0, v_1] \sim [v_1, v_3]$ and $[v_0, v_2] \sim [v_2, v_3]$ deformation retracts onto a Klein bottle. Also, find other pairs of identifications of edges that produce Δ-complexes deformation retracting onto a torus, a 2-sphere, and $\mathbb{R}P^2$.

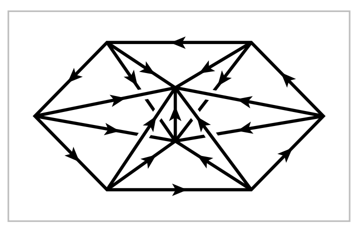
3. Construct a Δ-complex structure on $\mathbb{R}P^n$ as a quotient of a Δ-complex structure on S^n having vertices the two vectors of length 1 along each coordinate axis in $\mathbb{R}^{n+1}$.

4. Compute the simplicial homology groups of the triangular parachute obtained from Δ^2 by identifying its three vertices to a single point.

5. Compute the simplicial homology groups of the Klein bottle using the Δ-complex structure described at the beginning of this section.

6. Compute the simplicial homology groups of the Δ-complex obtained from $n+1$ 2-simplices $\Delta_0^2, \cdots, \Delta_n^2$ by identifying all three edges of Δ_0^2 to a single edge, and for $i > 0$ identifying the edges $[v_0, v_1]$ and $[v_1, v_2]$ of Δ_i^2 to a single edge and the edge $[v_0, v_2]$ to the edge $[v_0, v_1]$ of Δ_{i-1}^2.

7. Find a way of identifying pairs of faces of Δ^3 to produce a Δ-complex structure on S^3 having a single 3-simplex, and compute the simplicial homology groups of this Δ-complex.



8. Construct a 3-dimensional Δ-complex X from n tetrahedra $T_1, \cdots, T_n$ by the following two steps. First arrange the tetrahedra in a cyclic pattern as in the figure, so that each T_i shares a common vertical face with its two neighbors T_{i-1} and T_{i+1}, subscripts being taken mod n. Then identify the bottom face of T_i with the top face of T_{i+1} for each i. Show the simplicial homology groups of X in dimensions 0, 1, 2, 3 are $\mathbb{Z}$, $\mathbb{Z}_n$, 0, $\mathbb{Z}$, respectively. [The space X is an example of a *lens space*; see Example 2.43 for the general case.]

9. Compute the homology groups of the Δ-complex X obtained from Δ^n by identifying all faces of the same dimension. Thus X has a single k-simplex for each $k \le n$.

10. (a) Show the quotient space of a finite collection of disjoint 2-simplices obtained by identifying pairs of edges is always a surface, locally homeomorphic to $\mathbb{R}^2$.
(b) Show the edges can always be oriented so as to define a Δ-complex structure on the quotient surface. [This is more difficult.]

11. Show that if A is a retract of X then the map $H_n(A) \to H_n(X)$ induced by the inclusion $A \subset X$ is injective.

12. Show that chain homotopy of chain maps is an equivalence relation.

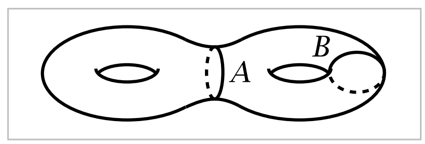
13. Verify that $f \simeq g$ implies $f_* = g_*$ for induced homomorphisms of reduced homology groups.

14. Determine whether there exists a short exact sequence $0 \to \mathbb{Z}_4 \to \mathbb{Z}_8 \oplus \mathbb{Z}_2 \to \mathbb{Z}_4 \to 0$. More generally, determine which abelian groups A fit into a short exact sequence $0 \to \mathbb{Z}_{p^m} \to A \to \mathbb{Z}_{p^n} \to 0$ with p prime. What about the case of short exact sequences $0 \to \mathbb{Z} \to A \to \mathbb{Z}_n \to 0$?

15. For an exact sequence $A \to B \to C \to D \to E$ show that $C = 0$ iff the map $A \to B$ is surjective and $D \to E$ is injective. Hence for a pair of spaces (X, A), the inclusion $A \hookrightarrow X$ induces isomorphisms on all homology groups iff $H_n(X, A) = 0$ for all n.

16. (a) Show that $H_0(X, A) = 0$ iff A meets each path-component of X.
(b) Show that $H_1(X, A) = 0$ iff $H_1(A) \to H_1(X)$ is surjective and each path-component of X contains at most one path-component of A.

17. (a) Compute the homology groups $H_n(X, A)$ when X is S^2 or $S^1 \times S^1$ and A is a finite set of points in X.
(b) Compute the groups $H_n(X, A)$ and $H_n(X, B)$ for X a closed orientable surface of genus two with A and B the circles shown. [What are X/A and X/B?]



18. Show that for the subspace $\mathbb{Q} \subset \mathbb{R}$, the relative homology group $H_1(\mathbb{R}, \mathbb{Q})$ is free abelian and find a basis.

19. Compute the homology groups of the subspace of $I \times I$ consisting of the four boundary edges plus all points in the interior whose first coordinate is rational.

20. Show that $\widetilde{H}_n(X) \approx \widetilde{H}_{n+1}(SX)$ for all n, where SX is the suspension of X. More generally, thinking of SX as the union of two cones CX with their bases identified, compute the reduced homology groups of the union of any finite number of cones CX with their bases identified.

21. Making the preceding problem more concrete, construct explicit chain maps $s : C_n(X) \to C_{n+1}(SX)$ inducing isomorphisms $\widetilde{H}_n(X) \to \widetilde{H}_{n+1}(SX)$.

22. Prove by induction on dimension the following facts about the homology of a finite-dimensional CW complex X, using the observation that X^n/X^{n-1} is a wedge sum of n-spheres:
(a) If X has dimension n then $H_i(X) = 0$ for $i > n$ and $H_n(X)$ is free.
(b) $H_n(X)$ is free with basis in bijective correspondence with the n-cells if there are no cells of dimension $n-1$ or $n+1$.
(c) If X has k n-cells, then $H_n(X)$ is generated by at most k elements.

23. Show that the second barycentric subdivision of a Δ-complex is a simplicial complex. Namely, show that the first barycentric subdivision produces a Δ-complex with the property that each simplex has all its vertices distinct, then show that for a Δ-complex with this property, barycentric subdivision produces a simplicial complex.

24. Show that each n-simplex in the barycentric subdivision of Δ^n is defined by n inequalities $t_{i_0} \le t_{i_1} \le \cdots \le t_{i_n}$ in its barycentric coordinates, where $(i_0, \cdots, i_n)$ is a permutation of $(0, \cdots, n)$.

25. Find an explicit, noninductive formula for the barycentric subdivision operator $S : C_n(X) \to C_n(X)$.

26. Show that $H_1(X, A)$ is not isomorphic to $\widetilde{H}_1(X/A)$ if $X = [0,1]$ and A is the sequence $1, 1/2, 1/3, \cdots$ together with its limit 0. [See Example 1.25.]

27. Let $f : (X, A) \to (Y, B)$ be a map such that both $f : X \to Y$ and the restriction $f : A \to B$ are homotopy equivalences.
(a) Show that $f_* : H_n(X, A) \to H_n(Y, B)$ is an isomorphism for all n.
(b) For the case of the inclusion $f : (D^n, S^{n-1}) \hookrightarrow (D^n, D^n - \{0\})$, show that f is not a homotopy equivalence of pairs — there is no $g : (D^n, D^n - \{0\}) \to (D^n, S^{n-1})$ such that fg and gf are homotopic to the identity through maps of pairs. [Observe that a homotopy equivalence of pairs $(X, A) \to (Y, B)$ is also a homotopy equivalence for the pairs obtained by replacing A and B by their closures.]

28. Let X be the cone on the 1-skeleton of Δ^3, the union of all line segments joining points in the six edges of Δ^3 to the barycenter of Δ^3. Compute the local homology groups $H_n(X, X - \{x\})$ for all $x \in X$. Define ∂X to be the subspace of points x such that $H_n(X, X - \{x\}) = 0$ for all n, and compute the local homology groups $H_n(\partial X, \partial X - \{x\})$. Use these calculations to determine which subsets $A \subset X$ have the property that $f(A) \subset A$ for all homeomorphisms $f : X \to X$.

29. Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups in all dimensions, but their universal covering spaces do not.

30. In each of the following commutative diagrams assume that all maps but one are isomorphisms. Show that the remaining map must be an isomorphism as well.

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & B \\ & C \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[d] & B \\ C \arrow[r] & D \arrow[u] \end{tikzcd}$$

31. Using the notation of the five-lemma, give an example where the maps α, β, δ, and ε are zero but γ is nonzero. This can be done with short exact sequences in which all the groups are either $\mathbb{Z}$ or 0.

2.2 Computations and Applications

Now that the basic properties of homology have been established, we can begin to move a little more freely. Our first topic, exploiting the calculation of $H_n(S^n)$, is Brouwer's notion of degree for maps $S^n \to S^n$. Historically, Brouwer's introduction of this concept in the years 1910–12 preceded the rigorous development of homology, so his definition was rather different, using the technique of simplicial approximation which we explain in §2.C. The later definition in terms of homology is certainly more elegant, though perhaps with some loss of geometric intuition. More in the spirit of Brouwer's definition is a third approach using differential topology, presented very lucidly in [Milnor 1965].

Degree

For a map $f : S^n \to S^n$ with $n > 0$, the induced map $f_* : H_n(S^n) \to H_n(S^n)$ is a homomorphism from an infinite cyclic group to itself and so must be of the form $f_*(\alpha) = d\alpha$ for some integer d depending only on f. This integer is called the **degree** of f, with the notation $\deg f$. Here are some basic properties of degree:

(a) $\deg \mathbb{1} = 1$, since $\mathbb{1}_* = \mathbb{1}$.

(b) $\deg f = 0$ if f is not surjective. For if we choose a point $x_0 \in S^n - f(S^n)$ then f can be factored as a composition $S^n \to S^n - \{x_0\} \hookrightarrow S^n$ and $H_n(S^n - \{x_0\}) = 0$ since $S^n - \{x_0\}$ is contractible. Hence $f_* = 0$.

(c) If $f \simeq g$ then $\deg f = \deg g$ since $f_* = g_*$. The converse statement, that $f \simeq g$ if $\deg f = \deg g$, is a fundamental theorem of Hopf from around 1925 which we prove in Corollary 4.25.

(d) $\deg fg = \deg f \deg g$, since $(fg)_* = f_* g_*$. As a consequence, $\deg f = \pm 1$ if f is a homotopy equivalence since $fg \simeq \mathbb{1}$ implies $\deg f \deg g = \deg \mathbb{1} = 1$.

(e) $\deg f = -1$ if f is a reflection of S^n, fixing the points in a subsphere S^{n-1} and interchanging the two complementary hemispheres. For we can give S^n a Δ-complex structure with these two hemispheres as its two n-simplices Δ_1^n and Δ_2^n, and the n-chain $\Delta_1^n - \Delta_2^n$ represents a generator of $H_n(S^n)$ as we saw in Example 2.23, so the reflection interchanging Δ_1^n and Δ_2^n sends this generator to its negative.

(f) The antipodal map $-\mathbb{1} : S^n \to S^n$, $x \mapsto -x$, has degree $(-1)^{n+1}$ since it is the composition of $n+1$ reflections, each changing the sign of one coordinate in $\mathbb{R}^{n+1}$.

(g) If $f : S^n \to S^n$ has no fixed points then $\deg f = (-1)^{n+1}$. For if $f(x) \neq x$ then the line segment from $f(x)$ to $-x$, defined by $t \mapsto (1-t)f(x) - tx$ for $0 \le t \le 1$, does not pass through the origin. Hence if f has no fixed points, the formula $f_t(x) = [(1-t)f(x) - tx]/|(1-t)f(x) - tx|$ defines a homotopy from f to the antipodal map. Note that the antipodal map has no fixed points, so the fact that maps without fixed points are homotopic to the antipodal map is a sort of converse statement.

Here is an interesting application of degree:

Theorem 2.28. *S^n has a continuous field of nonzero tangent vectors iff n is odd.*

Proof: Suppose $x \mapsto v(x)$ is a tangent vector field on S^n, assigning to a vector $x \in S^n$ the vector $v(x)$ tangent to S^n at x. Regarding $v(x)$ as a vector at the origin instead of at x, tangency just means that x and $v(x)$ are orthogonal in $\mathbb{R}^{n+1}$. If $v(x) \neq 0$ for all x, we may normalize so that $|v(x)| = 1$ for all x by replacing $v(x)$ by $v(x)/|v(x)|$. Assuming this has been done, the vectors $(\cos t)x + (\sin t)v(x)$ lie in the unit circle in the plane spanned by x and $v(x)$. Letting t go from 0 to π, we obtain a homotopy $f_t(x) = (\cos t)x + (\sin t)v(x)$ from the identity map of S^n to the antipodal map $-\mathbb{1}$. This implies that $\deg(-\mathbb{1}) = \deg \mathbb{1}$, hence $(-1)^{n+1} = 1$ and n must be odd.

Conversely, if n is odd, say $n = 2k-1$, we can define $v(x_1, x_2, \cdots, x_{2k-1}, x_{2k}) = (-x_2, x_1, \cdots, -x_{2k}, x_{2k-1})$. Then $v(x)$ is orthogonal to x, so v is a tangent vector field on S^n, and $|v(x)| = 1$ for all $x \in S^n$. $\square$

For the much more difficult problem of finding the maximum number of tangent vector fields on S^n that are linearly independent at each point, see [VBKT] or [Husemoller 1966].

Another nice application of degree, giving a partial answer to a question raised in Example 1.43, is the following result:

Proposition 2.29. *$\mathbb{Z}_2$ is the only nontrivial group that can act freely on S^n if n is even.*

Recall that an action of a group G on a space X is a homomorphism from G to the group $\text{Homeo}(X)$ of homeomorphisms $X \to X$, and the action is free if the homeomorphism corresponding to each nontrivial element of G has no fixed points. In the case of S^n, the antipodal map $x \mapsto -x$ generates a free action of $\mathbb{Z}_2$.

Proof: Since homeomorphisms have degree ± 1, an action of a group G on S^n determines a degree function $d : G \to \{\pm 1\}$. This is a homomorphism since $\deg fg = \deg f \deg g$. If the action is free, d sends each nontrivial element of G to $(-1)^{n+1}$ by property (g) above. Thus when n is even, d has trivial kernel, so $G \subset \mathbb{Z}_2$. $\square$

Next we describe a technique for computing degrees which can be applied to most maps that arise in practice. Suppose $f : S^n \to S^n$, $n > 0$, has the property that for some point $y \in S^n$, the preimage $f^{-1}(y)$ consists of only finitely many points, say

$x_1, \cdots, x_m$. Let $U_1, \cdots, U_m$ be disjoint neighborhoods of these points, mapped by f into a neighborhood V of y. Then $f(U_i - x_i) \subset V - y$ for each i, and we have a diagram

$$\begin{tikzcd}
& H_n(U_i, U_i - x_i) \arrow[dl, "\approx"'] \arrow[r, "f_*"] \arrow[d, "k_i"] & H_n(V, V - y) \arrow[d, "\approx"] \\
H_n(S^n, S^n - x_i) & H_n(S^n, S^n - f^{-1}(y)) \arrow[l, "p_i"'] \arrow[r, "f_*"] & H_n(S^n, S^n - y) \\
& H_n(S^n) \arrow[ul, "\approx"] \arrow[u, "j"'] \arrow[r, "f_*"] & H_n(S^n) \arrow[u, "\approx"']
\end{tikzcd}$$

where all the maps are the obvious ones, and in particular k_i and p_i are induced by inclusions, so the triangles and squares commute. The two isomorphisms in the upper half of the diagram come from excision, while the lower two isomorphisms come from exact sequences of pairs. Via these four isomorphisms, the top two groups in the diagram can be identified with $H_n(S^n) \approx \mathbb{Z}$, and the top homomorphism f_* becomes multiplication by an integer called the **local degree** of f at x_i, written $\deg f \,|\, x_i$.

For example, if f is a homeomorphism, then y can be any point and there is only one corresponding x_i, so all the maps in the diagram are isomorphisms and $\deg f \,|\, x_i = \deg f = \pm 1$. More generally, if f maps each U_i homeomorphically onto V, then $\deg f \,|\, x_i = \pm 1$ for each i. This situation occurs quite often in applications, and it is usually not hard to determine the correct signs.

Here is the formula that reduces degree calculations to computing local degrees:

Proposition 2.30. $\deg f = \sum_i \deg f \,|\, x_i$.

Proof: By excision, the central term $H_n(S^n, S^n - f^{-1}(y))$ in the preceding diagram is the direct sum of the groups $H_n(U_i, U_i - x_i) \approx \mathbb{Z}$, with k_i the inclusion of the i^{th} summand. The map p_i is projection onto the i^{th} summand since the upper triangle commutes and $p_i k_j = 0$ for $j \neq i$, as $p_i k_j$ factors through $H_n(U_j, U_j) = 0$. Identifying the outer groups in the diagram with $\mathbb{Z}$ as before, commutativity of the lower triangle says that $p_i j(1) = 1$, hence $j(1) = (1, \cdots, 1) = \sum_i k_i(1)$. Commutativity of the upper square says that the middle f_* takes $k_i(1)$ to $\deg f \,|\, x_i$, hence the sum $\sum_i k_i(1) = j(1)$ is taken to $\sum_i \deg f \,|\, x_i$. Commutativity of the lower square then gives the formula $\deg f = \sum_i \deg f \,|\, x_i$. $\square$

Example 2.31. We can use this result to construct a map $S^n \to S^n$ of any given degree, for each $n \geq 1$. Let $q : S^n \to \bigvee_k S^n$ be the quotient map obtained by collapsing the complement of k disjoint open balls B_i in S^n to a point, and let $p : \bigvee_k S^n \to S^n$ identify all the summands to a single sphere. Consider the composition $f = pq$. For almost all $y \in S^n$ we have $f^{-1}(y)$ consisting of one point x_i in each B_i. The local degree of f at x_i is ± 1 since f is a homeomorphism near x_i. By precomposing p with reflections of the summands of $\bigvee_k S^n$ if necessary, we can make each local degree either $+1$ or -1, whichever we wish. Thus we can produce a map $S^n \to S^n$ of degree $\pm k$.

Example 2.32. In the case of S^1, the map $f(z) = z^k$, where we view S^1 as the unit circle in $\mathbb{C}$, has degree k. This is evident in the case $k = 0$ since f is then constant. The case $k < 0$ reduces to the case $k > 0$ by composing with $z \mapsto z^{-1}$, which is a reflection, of degree -1. To compute the degree when $k > 0$, observe first that for any $y \in S^1$, $f^{-1}(y)$ consists of k points $x_1, \cdots, x_k$ near each of which f is a local homeomorphism, stretching a circular arc by a factor of k. This local stretching can be eliminated by a deformation of f near x_i that does not change local degree, so the local degree at x_i is the same as for a rotation of S^1. A rotation is a homeomorphism so its local degree at any point equals its global degree, which is $+1$ since a rotation is homotopic to the identity. Hence $\deg f\,|\,x_i = 1$ and $\deg f = k$.

Another way of obtaining a map $S^n \to S^n$ of degree k is to take a repeated suspension of the map $z \mapsto z^k$ in Example 2.32, since suspension preserves degree:

Proposition 2.33. $\deg Sf = \deg f$, *where* $Sf : S^{n+1} \to S^{n+1}$ *is the suspension of the map* $f : S^n \to S^n$.

Proof: Let CS^n denote the cone $(S^n \times I)/(S^n \times 1)$ with base $S^n = S^n \times 0 \subset CS^n$, so CS^n/S^n is the suspension of S^n. The map f induces $Cf : (CS^n, S^n) \to (CS^n, S^n)$ with quotient Sf. The naturality of the boundary maps in the long exact sequence of the pair (CS^n, S^n) then gives commutativity of the diagram at the right. Hence if f_* is multiplication by d, so is Sf_*. □

$$\begin{tikzcd} \widetilde{H}_{n+1}(S^{n+1}) \arrow[r, "\partial", "\approx"'] \arrow[d, "Sf_*"] & \widetilde{H}_n(S^n) \arrow[d, "f_*"] \\ \widetilde{H}_{n+1}(S^{n+1}) \arrow[r, "\partial", "\approx"'] & \widetilde{H}_n(S^n) \end{tikzcd}$$

Note that for $f : S^n \to S^n$, the suspension Sf maps only one point to each of the two 'poles' of S^{n+1}. This implies that the local degree of Sf at each pole must equal the global degree of Sf. Thus the local degree of a map $S^n \to S^n$ can be any integer if $n \ge 2$, just as the degree itself can be any integer when $n \ge 1$.

Cellular Homology

Cellular homology is a very efficient tool for computing the homology groups of CW complexes, based on degree calculations. Before giving the definition of cellular homology, we first establish a few preliminary facts:

Lemma 2.34. *If X is a CW complex, then*:

(a) $H_k(X^n, X^{n-1})$ *is zero for* $k \ne n$ *and is free abelian for* $k = n$, *with a basis in one-to-one correspondence with the* n*-cells of* X.

(b) $H_k(X^n) = 0$ *for* $k > n$. *In particular, if X is finite-dimensional then* $H_k(X) = 0$ *for* $k > \dim X$.

(c) *The map* $H_k(X^n) \to H_k(X)$ *induced by the inclusion* $X^n \hookrightarrow X$ *is an isomorphism for* $k < n$ *and surjective for* $k = n$.

Proof: Statement (a) follows immediately from the observation that (X^n, X^{n-1}) is a good pair and X^n/X^{n-1} is a wedge sum of n-spheres, one for each n-cell of X. Here we are using Proposition 2.22 and Corollary 2.25. Next consider the following part of the long exact sequence of the pair (X^n, X^{n-1}):

$$H_{k+1}(X^n, X^{n-1}) \longrightarrow H_k(X^{n-1}) \longrightarrow H_k(X^n) \longrightarrow H_k(X^n, X^{n-1})$$

If $k \neq n$ the last term is zero by part (a) so the middle map is surjective, while if $k \neq n-1$ then the first term is zero so the middle map is injective. Now look at the inclusion-induced homomorphisms

$$H_k(X^0) \longrightarrow H_k(X^1) \longrightarrow \cdots \longrightarrow H_k(X^{k-1}) \longrightarrow H_k(X^k) \longrightarrow H_k(X^{k+1}) \longrightarrow \cdots$$

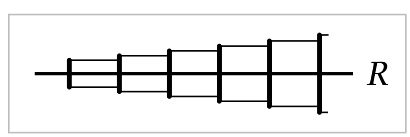
By what we have just shown these are all isomorphisms except that the map to $H_k(X^k)$ may not be surjective and the map from $H_k(X^k)$ may not be injective. The first part of the sequence then gives statement (b) since $H_k(X^0) = 0$ when $k > 0$. Also, the last part of the sequence gives (c) when X is finite-dimensional.

The proof of (c) when X is infinite-dimensional requires more work, and this can be done in two different ways. The more direct approach is to descend to the chain level and use the fact that a singular chain in X has compact image, hence meets only finitely many cells of X by Proposition A.1 in the Appendix. Thus each chain lies in a finite skeleton X^m. So a k-cycle in X is a cycle in some X^m, and then by the finite-dimensional case of (c), the cycle is homologous to a cycle in X^n if $n \geq k$, so $H_k(X^n) \to H_k(X)$ is surjective. Similarly for injectivity, if a k-cycle in X^n bounds a chain in X, this chain lies in some X^m with $m \geq n$, so by the finite-dimensional case the cycle bounds a chain in X^n if $n > k$.

The other approach is more general. From the long exact sequence of the pair (X, X^n) it suffices to show $H_k(X, X^n) = 0$ for $k \leq n$. Since $H_k(X, X^n) \approx \widetilde{H}_k(X/X^n)$, this reduces the problem to showing:

$(*)$ $\widetilde{H}_k(X) = 0$ for $k \leq n$ if the n-skeleton of X is a point.

When X is finite-dimensional, $(*)$ is immediate from the finite-dimensional case of (c) which we have already shown. It will suffice therefore to reduce the infinite-dimensional case to the finite-dimensional case. This reduction will be achieved by stretching X out to a complex that is at least locally finite-dimensional, using a special case of the 'mapping telescope' construction described in greater generality in §3.F.



Consider $X \times [0, \infty)$ with its product cell structure, where we give $[0, \infty)$ the cell structure with the integer points as 0-cells. Let $T = \bigcup_i X^i \times [i, \infty)$, a subcomplex of $X \times [0, \infty)$. The figure shows a schematic picture of T with $[0, \infty)$ in the horizontal direction and the subcomplexes $X^i \times [i, i+1]$ as rectangles whose size increases with i since $X^i \subset X^{i+1}$. The line labeled R can be ignored for now. We claim that $T \simeq X$, hence $H_k(X) \approx H_k(T)$ for all k. Since X is a deformation retract of $X \times [0, \infty)$, it suffices to show that $X \times [0, \infty)$ also deformation retracts onto T. Let $Y_i = T \cup (X \times [i, \infty))$. Then Y_i deformation retracts onto Y_{i+1} since $X \times [i, i+1]$ deformation retracts onto $X^i \times [i, i+1] \cup X \times \{i+1\}$ by Proposition 0.16. If we perform the deformation retraction of Y_i onto Y_{i+1} during the t-interval $[1-1/2^i, 1-1/2^{i+1}]$, then this gives a deformation retraction f_t of $X\times[0,\infty)$ onto T, with points in $X^i\times[0,\infty)$ stationary under f_t for $t \ge 1-1/2^{i+1}$. Continuity follows from the fact that CW complexes have the weak topology with respect to their skeleta, so a map is continuous if its restriction to each skeleton is continuous.

Recalling that X^0 is a point, let $R \subset T$ be the ray $X^0\times[0,\infty)$ and let $Z \subset T$ be the union of this ray with all the subcomplexes $X^i\times\{i\}$. Then Z/R is homeomorphic to $\bigvee_i X^i$, a wedge sum of finite-dimensional complexes with n-skeleton a point, so the finite-dimensional case of $(*)$ together with Corollary 2.25 describing the homology of wedge sums implies that $\widetilde{H}_k(Z/R) = 0$ for $k \le n$. The same is therefore true for Z, from the long exact sequence of the pair (Z,R), since R is contractible. Similarly, T/Z is a wedge sum of finite-dimensional complexes with $(n+1)$-skeleton a point, since if we first collapse each subcomplex $X^i\times\{i\}$ of T to a point, we obtain the infinite sequence of suspensions SX^i 'skewered' along the ray R, and then if we collapse R to a point we obtain $\bigvee_i \Sigma X^i$ where ΣX^i is the reduced suspension of X^i, obtained from SX^i by collapsing the line segment $X^0\times[i,i+1]$ to a point, so ΣX^i has $(n+1)$-skeleton a point. Thus $\widetilde{H}_k(T/Z) = 0$ for $k \le n+1$. The long exact sequence of the pair (T,Z) then implies that $\widetilde{H}_k(T) = 0$ for $k \le n$, and we have proved $(*)$. $\square$

Let X be a CW complex. Using Lemma 2.34, portions of the long exact sequences for the pairs (X^{n+1},X^n), (X^n,X^{n-1}), and (X^{n-1},X^{n-2}) fit into a diagram

$$\begin{tikzcd}
 & & & & 0 & & \\
 & 0 \arrow[dr] & & H_n(X^{n+1}) \approx H_n(X) \arrow[ur] & & & \\
 & & H_n(X^n) \arrow[ur] \arrow[dr, "j_n"] & & & & \\
\cdots \arrow[r] & H_{n+1}(X^{n+1},X^n) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"] & & H_n(X^n,X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"'] & & H_{n-1}(X^{n-1},X^{n-2}) \arrow[r] & \cdots \\
 & & & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"'] & & \\
 & & & 0 \arrow[ur] & & &
\end{tikzcd}$$

where d_{n+1} and d_n are defined as the compositions $j_n\partial_{n+1}$ and $j_{n-1}\partial_n$, which are just 'relativizations' of the boundary maps ∂_{n+1} and ∂_n. The composition d_nd_{n+1} includes two successive maps in one of the exact sequences, hence is zero. Thus the horizontal row in the diagram is a chain complex, called the **cellular chain complex** of X since $H_n(X^n,X^{n-1})$ is free with basis in one-to-one correspondence with the n-cells of X, so one can think of elements of $H_n(X^n,X^{n-1})$ as linear combinations of n-cells of X. The homology groups of this cellular chain complex are called the **cellular homology groups** of X. Temporarily we denote them $H_n^{CW}(X)$.

Theorem 2.35. $H_n^{CW}(X) \approx H_n(X)$.

Proof: From the diagram above, $H_n(X)$ can be identified with $H_n(X^n)/\operatorname{Im}\partial_{n+1}$. Since j_n is injective, it maps $\operatorname{Im}\partial_{n+1}$ isomorphically onto $\operatorname{Im}(j_n\partial_{n+1}) = \operatorname{Im} d_{n+1}$ and $H_n(X^n)$ isomorphically onto $\operatorname{Im} j_n = \operatorname{Ker}\partial_n$. Since j_{n-1} is injective, $\operatorname{Ker}\partial_n = \operatorname{Ker} d_n$. Thus j_n induces an isomorphism of the quotient $H_n(X^n)/\operatorname{Im}\partial_{n+1}$ onto $\operatorname{Ker} d_n/\operatorname{Im} d_{n+1}$. □

Here are a few immediate applications:

(i) $H_n(X) = 0$ if X is a CW complex with no n-cells.

(ii) More generally, if X is a CW complex with k n-cells, then $H_n(X)$ is generated by at most k elements. For since $H_n(X^n, X^{n-1})$ is free abelian on k generators, the subgroup $\operatorname{Ker} d_n$ must be generated by at most k elements, hence also the quotient $\operatorname{Ker} d_n/\operatorname{Im} d_{n+1}$.

(iii) If X is a CW complex having no two of its cells in adjacent dimensions, then $H_n(X)$ is free abelian with basis in one-to-one correspondence with the n-cells of X. This is because the cellular boundary maps d_n are automatically zero in this case.

This last observation applies for example to $\mathbb{C}\mathrm{P}^n$, which has a CW structure with one cell of each even dimension $2k \le 2n$ as we saw in Example 0.6. Thus

$$H_i(\mathbb{C}\mathrm{P}^n) \approx \begin{cases} \mathbb{Z} & \text{for } i = 0, 2, 4, \cdots, 2n \\ 0 & \text{otherwise} \end{cases}$$

Another simple example is $S^n \times S^n$ with $n > 1$, using the product CW structure consisting of a 0-cell, two n-cells, and a $2n$-cell.

It is possible to prove the statements (i)–(iii) for finite-dimensional CW complexes by induction on the dimension, without using cellular homology but only the basic results from the previous section. However, the viewpoint of cellular homology makes (i)–(iii) quite transparent.

Next we describe how the cellular boundary maps d_n can be computed. When $n = 1$ this is easy since the boundary map $d_1 : H_1(X^1, X^0) \to H_0(X^0)$ is the same as the simplicial boundary map $\Delta_1(X) \to \Delta_0(X)$. In case X is connected and has only one 0-cell, then d_1 must be 0, otherwise $H_0(X)$ would not be $\mathbb{Z}$. When $n > 1$ we will show that d_n can be computed in terms of degrees:

Cellular Boundary Formula. $d_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta} e^{n-1}_\beta$ *where* $d_{\alpha\beta}$ *is the degree of the map* $S^{n-1}_\alpha \to X^{n-1} \to S^{n-1}_\beta$ *that is the composition of the attaching map of* e^n_α *with the quotient map collapsing* $X^{n-1} - e^{n-1}_\beta$ *to a point.*

Here we are identifying the cells e^n_α and e^{n-1}_β with generators of the corresponding summands of the cellular chain groups. The summation in the formula contains only finitely many terms since the attaching map of e^n_α has compact image, so this image meets only finitely many cells e^{n-1}_β.

To derive the cellular boundary formula, consider the commutative diagram

$$\begin{tikzcd}
H_n(D^n_\alpha,\partial D^n_\alpha) \arrow[r, "\partial", "\approx"'] \arrow[d, "\Phi_{\alpha*}"] & \widetilde{H}_{n-1}(\partial D^n_\alpha) \arrow[r, "\Delta_{\alpha\beta*}"] \arrow[d, "\varphi_{\alpha*}"] & \widetilde{H}_{n-1}(S^{n-1}_\beta) \\
H_n(X^n,X^{n-1}) \arrow[r, "\partial_n"] \arrow[dr, "d_n"'] & \widetilde{H}_{n-1}(X^{n-1}) \arrow[r, "q_*"] \arrow[d, "j_{n-1}"] & \widetilde{H}_{n-1}(X^{n-1}/X^{n-2}) \arrow[u, "q_{\beta*}"'] \arrow[d, "\approx"] \\
& H_{n-1}(X^{n-1},X^{n-2}) \arrow[r, "\approx"] & H_{n-1}(X^{n-1}/X^{n-2},X^{n-2}/X^{n-2})
\end{tikzcd}$$

where:

- Φ_α is the characteristic map of the cell e^n_α and φ_α is its attaching map.
- $q : X^{n-1} \to X^{n-1}/X^{n-2}$ is the quotient map.
- $q_\beta : X^{n-1}/X^{n-2} \to S^{n-1}_\beta$ collapses the complement of the cell e^{n-1}_β to a point, the resulting quotient sphere being identified with $S^{n-1}_\beta = D^{n-1}_\beta/\partial D^{n-1}_\beta$ via the characteristic map Φ_β.
- $\Delta_{\alpha\beta} : \partial D^n_\alpha \to S^{n-1}_\beta$ is the composition $q_\beta q \varphi_\alpha$, in other words, the attaching map of e^n_α followed by the quotient map $X^{n-1} \to S^{n-1}_\beta$ collapsing the complement of e^{n-1}_β in X^{n-1} to a point.

The map $\Phi_{\alpha*}$ takes a chosen generator $[D^n_\alpha] \in H_n(D^n_\alpha, \partial D^n_\alpha)$ to a generator of the $\mathbb{Z}$ summand of $H_n(X^n, X^{n-1})$ corresponding to e^n_α. Letting e^n_α denote this generator, commutativity of the left half of the diagram then gives $d_n(e^n_\alpha) = j_{n-1}\varphi_{\alpha*}\partial[D^n_\alpha]$. In terms of the basis for $H_{n-1}(X^{n-1}, X^{n-2})$ corresponding to the cells e^{n-1}_β, the map $q_{\beta*}$ is the projection of $\widetilde{H}_{n-1}(X^{n-1}/X^{n-2})$ onto its $\mathbb{Z}$ summand corresponding to e^{n-1}_β. Commutativity of the diagram then yields the formula for d_n given above.

Example 2.36. Let M_g be the closed orientable surface of genus g with its usual CW structure consisting of one 0-cell, $2g$ 1-cells, and one 2-cell attached by the product of commutators $[a_1, b_1] \cdots [a_g, b_g]$. The associated cellular chain complex is

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{2g} \xrightarrow{d_1} \mathbb{Z} \longrightarrow 0$$

As observed above, d_1 must be 0 since there is only one 0-cell. Also, d_2 is 0 because each a_i or b_i appears with its inverse in $[a_1, b_1] \cdots [a_g, b_g]$, so the maps $\Delta_{\alpha\beta}$ are homotopic to constant maps. Since d_1 and d_2 are both zero, the homology groups of M_g are the same as the cellular chain groups, namely, $\mathbb{Z}$ in dimensions 0 and 2, and $\mathbb{Z}^{2g}$ in dimension 1.

Example 2.37. The closed nonorientable surface N_g of genus g has a cell structure with one 0-cell, g 1-cells, and one 2-cell attached by the word $a_1^2 a_2^2 \cdots a_g^2$. Again $d_1 = 0$, and $d_2 : \mathbb{Z} \to \mathbb{Z}^g$ is specified by the equation $d_2(1) = (2, \cdots, 2)$ since each a_i appears in the attaching word of the 2-cell with total exponent 2, which means that each $\Delta_{\alpha\beta}$ is homotopic to the map $z \mapsto z^2$, of degree 2. Since $d_2(1) = (2, \cdots, 2)$, we have d_2 injective and hence $H_2(N_g) = 0$. If we change the basis for $\mathbb{Z}^g$ by replacing the last standard basis element $(0, \cdots, 0, 1)$ by $(1, \cdots, 1)$, we see that $H_1(N_g) \approx \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$.

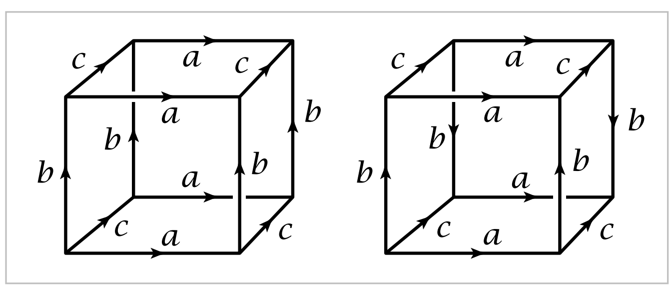
These two examples illustrate the general fact that the orientability of a closed connected manifold M of dimension n is detected by $H_n(M)$, which is $\mathbb{Z}$ if M is orientable and 0 otherwise. This is shown in Theorem 3.26.

Example 2.38: An Acyclic Space. Let X be obtained from $S^1 \vee S^1$ by attaching two 2-cells by the words a^5b^{-3} and $b^3(ab)^{-2}$. Then $d_2:\mathbb{Z}^2 \to \mathbb{Z}^2$ has matrix $\begin{pmatrix} 5 & -2 \\ -3 & 1 \end{pmatrix}$, with the two columns coming from abelianizing a^5b^{-3} and $b^3(ab)^{-2}$ to $5a - 3b$ and $-2a + b$, in additive notation. The matrix has determinant -1, so d_2 is an isomorphism and $\widetilde{H}_i(X) = 0$ for all i. Such a space X is called **acyclic.**

We can see that this acyclic space is not contractible by considering $\pi_1(X)$, which has the presentation $\langle\, a, b \mid a^5b^{-3}, b^3(ab)^{-2}\,\rangle$. There is a nontrivial homomorphism from this group to the group G of rotational symmetries of a regular dodecahedron, sending a to the rotation ρ_a through angle $2\pi/5$ about the axis through the center of a pentagonal face, and b to the rotation ρ_b through angle $2\pi/3$ about the axis through a vertex of this face. The composition $\rho_a\rho_b$ is a rotation through angle π about the axis through the midpoint of an edge abutting this vertex. Thus the relations $a^5 = b^3 = (ab)^2$ defining $\pi_1(X)$ become $\rho_a^5 = \rho_b^3 = (\rho_a\rho_b)^2 = 1$ in G, which means there is a well-defined homomorphism $\rho:\pi_1(X) \to G$ sending a to ρ_a and b to ρ_b. It is not hard to see that G is generated by ρ_a and ρ_b, so ρ is surjective. With more work one can compute that the kernel of ρ is $\mathbb{Z}_2$, generated by the element $a^5 = b^3 = (ab)^2$, and this $\mathbb{Z}_2$ is in fact the center of $\pi_1(X)$. In particular, $\pi_1(X)$ has order 120 since G has order 60.

After these 2-dimensional examples, let us now move up to three dimensions, where we have the additional task of computing the cellular boundary map d_3.

Example 2.39. A 3-dimensional torus $T^3 = S^1 \times S^1 \times S^1$ can be constructed from a cube by identifying each pair of opposite square faces as in the first of the two figures. The second figure shows a slightly different pattern of identifications of opposite faces, with the front and back faces now identified via a rotation of the cube around a horizontal left-right axis. The space produced by these identifications is the product $K \times S^1$ of a Klein bottle and a circle. For both T^3 and $K \times S^1$ we have a CW structure with one 3-cell, three 2-cells, three 1-cells, and one 0-cell. The cellular chain complexes thus have the form



$$0 \longrightarrow \mathbb{Z} \xrightarrow{d_3} \mathbb{Z}^3 \xrightarrow{d_2} \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

In the case of the 3-torus T^3 the cellular boundary map d_2 is zero by the same calculation as for the 2-dimensional torus. We claim that d_3 is zero as well. This amounts to saying that the three maps $\Delta_{\alpha\beta}:S^2 \to S^2$ corresponding to the three 2-cells have degree zero. Each $\Delta_{\alpha\beta}$ maps the interiors of two opposite faces of the cube homeomorphically onto the complement of a point in the target S^2 and sends the remaining four faces to this point. Computing local degrees at the center points of the two opposite faces, we see that the local degree is $+1$ at one of these points and -1 at the other, since the restrictions of $\Delta_{\alpha\beta}$ to these two faces differ by a reflection of the boundary of the cube across the plane midway between them, and a reflection has degree -1. Since the cellular boundary maps are all zero, we deduce that $H_i(T^3)$ is $\mathbb{Z}$ for $i = 0, 3$, $\mathbb{Z}^3$ for $i = 1, 2$, and 0 for $i > 3$.

For $K\times S^1$, when we compute local degrees for the front and back faces we find that the degrees now have the same rather than opposite signs since the map $\Delta_{\alpha\beta}$ on these two faces differs not by a reflection but by a rotation of the boundary of the cube. The local degrees for the other faces are the same as before. Using the letters A, B, C to denote the 2-cells given by the faces orthogonal to the edges a, b, c, respectively, we have the boundary formulas $d_3e^3 = 2C$, $d_2A = 2b$, $d_2B = 0$, and $d_2C = 0$. It follows that $H_3(K\times S^1) = 0$, $H_2(K\times S^1) = \mathbb{Z}\oplus\mathbb{Z}_2$, and $H_1(K\times S^1) = \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}_2$.

Many more examples of a similar nature, quotients of a cube or other polyhedron with faces identified in some pattern, could be worked out in similar fashion. But let us instead turn to some higher-dimensional examples.

Example 2.40: Moore Spaces. Given an abelian group G and an integer $n \ge 1$, we will construct a CW complex X such that $H_n(X) \approx G$ and $\widetilde{H}_i(X) = 0$ for $i \ne n$. Such a space is called a **Moore space**, commonly written $M(G,n)$ to indicate the dependence on G and n. It is probably best for the definition of a Moore space to include the condition that $M(G,n)$ be simply-connected if $n > 1$. The spaces we construct will have this property.

As an easy special case, when $G = \mathbb{Z}_m$ we can take X to be S^n with a cell e^{n+1} attached by a map $S^n \to S^n$ of degree m. More generally, any finitely generated G can be realized by taking wedge sums of examples of this type for finite cyclic summands of G, together with copies of S^n for infinite cyclic summands of G.

In the general nonfinitely generated case let $F \to G$ be a homomorphism of a free abelian group F onto G, sending a basis for F onto some set of generators of G. The kernel K of this homomorphism is a subgroup of a free abelian group, hence is itself free abelian. Choose bases $\{x_\alpha\}$ for F and $\{y_\beta\}$ for K, and write $y_\beta = \sum_\alpha d_{\beta\alpha}x_\alpha$. Let $X^n = \bigvee_\alpha S^n_\alpha$, so $H_n(X^n) \approx F$ via Corollary 2.25. We will construct X from X^n by attaching cells e^{n+1}_β via maps $f_\beta : S^n \to X^n$ such that the composition of f_β with the projection onto the summand S^n_α has degree $d_{\beta\alpha}$. Then the cellular boundary map d_{n+1} will be the inclusion $K \hookrightarrow F$, hence X will have the desired homology groups.

The construction of f_β generalizes the construction in Example 2.31 of a map $S^n \to S^n$ of given degree. Namely, we can let f_β map the complement of $\sum_\alpha |d_{\beta\alpha}|$

disjoint balls in S^n to the 0-cell of X^n while sending $|d_{\beta\alpha}|$ of the balls onto the summand S^n_α by maps of degree $+1$ if $d_{\beta\alpha} > 0$, or degree -1 if $d_{\beta\alpha} < 0$.

Example 2.41. By taking a wedge sum of the Moore spaces constructed in the preceding example for varying n we obtain a connected CW complex with any prescribed sequence of homology groups in dimensions $1, 2, 3, \cdots$.

Example 2.42: Real Projective Space $\mathbb{R}P^n$. As we saw in Example 0.4, $\mathbb{R}P^n$ has a CW structure with one cell e^k in each dimension $k \le n$, and the attaching map for e^k is the 2-sheeted covering projection $\varphi : S^{k-1} \to \mathbb{R}P^{k-1}$. To compute the boundary map d_k we compute the degree of the composition $S^{k-1} \xrightarrow{\varphi} \mathbb{R}P^{k-1} \xrightarrow{q} \mathbb{R}P^{k-1}/\mathbb{R}P^{k-2} = S^{k-1}$, with q the quotient map. The map $q\varphi$ restricts to a homeomorphism from each component of $S^{k-1} - S^{k-2}$ onto $\mathbb{R}P^{k-1} - \mathbb{R}P^{k-2}$, and these two homeomorphisms are obtained from each other by precomposing with the antipodal map of S^{k-1}, which has degree $(-1)^k$. Hence $\deg q\varphi = \deg \mathbb{1} + \deg(-\mathbb{1}) = 1 + (-1)^k$, and so d_k is either 0 or multiplication by 2 according to whether k is odd or even. Thus the cellular chain complex for $\mathbb{R}P^n$ is

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0 \qquad \text{if } n \text{ is even}$$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0 \qquad \text{if } n \text{ is odd}$$

From this it follows that

$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & \text{for } k = 0 \text{ and for } k = n \text{ odd} \\ \mathbb{Z}_2 & \text{for } k \text{ odd, } 0 < k < n \\ 0 & \text{otherwise} \end{cases}$$

Example 2.43: Lens Spaces. This example is somewhat more complicated. Given an integer $m > 1$ and integers $\ell_1, \cdots, \ell_n$ relatively prime to m, define the **lens space** $L = L_m(\ell_1, \cdots, \ell_n)$ to be the orbit space $S^{2n-1}/\mathbb{Z}_m$ of the unit sphere $S^{2n-1} \subset \mathbb{C}^n$ with the action of $\mathbb{Z}_m$ generated by the rotation $\rho(z_1, \cdots, z_n) = (e^{2\pi i \ell_1/m} z_1, \cdots, e^{2\pi i \ell_n/m} z_n)$, rotating the j^{th} $\mathbb{C}$ factor of $\mathbb{C}^n$ by the angle $2\pi\ell_j/m$. In particular, when $m = 2$, ρ is the antipodal map, so $L = \mathbb{R}P^{2n-1}$ in this case. In the general case, the projection $S^{2n-1} \to L$ is a covering space since the action of $\mathbb{Z}_m$ on S^{2n-1} is free: Only the identity element fixes any point of S^{2n-1} since each point of S^{2n-1} has some coordinate z_j nonzero and then $e^{2\pi i k \ell_j/m} z_j \ne z_j$ for $0 < k < m$, as a result of the assumption that ℓ_j is relatively prime to m.

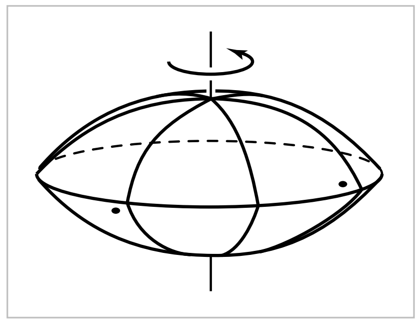
We shall construct a CW structure on L with one cell e^k for each $k \le 2n-1$ and show that the resulting cellular chain complex is

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

with boundary maps alternately 0 and multiplication by m. Hence

$$H_k(L_m(\ell_1, \cdots, \ell_n)) = \begin{cases} \mathbb{Z} & \text{for } k = 0, 2n-1 \\ \mathbb{Z}_m & \text{for } k \text{ odd, } 0 < k < 2n-1 \\ 0 & \text{otherwise} \end{cases}$$

To obtain the CW structure, first subdivide the unit circle C in the n^{th} $\mathbb{C}$ factor of $\mathbb{C}^n$ by taking the points $e^{2\pi ij/m} \in C$ as vertices, $j = 1, \cdots, m$. Joining the j^{th} vertex of C to the unit sphere $S^{2n-3} \subset \mathbb{C}^{n-1}$ by arcs of great circles in S^{2n-1} yields a $(2n-2)$-dimensional ball B_j^{2n-2} bounded by S^{2n-3}. Specifically, B_j^{2n-2} consists of the points $\cos\theta\,(0, \cdots, 0, e^{2\pi ij/m}) + \sin\theta\,(z_1, \cdots, z_{n-1}, 0)$ for $0 \le \theta \le \pi/2$. Similarly, joining the j^{th} edge of C to S^{2n-3} gives a ball B_j^{2n-1} bounded by B_j^{2n-2} and B_{j+1}^{2n-2}, subscripts being taken mod m. The rotation ρ carries S^{2n-3} to itself and rotates C by the angle $2\pi\ell_n/m$, hence ρ permutes the B_j^{2n-2}'s and the B_j^{2n-1}'s. A suitable power of ρ, namely ρ^r where $r\ell_n \equiv 1$ mod m, takes each B_j^{2n-2} and B_j^{2n-1} to the next one. Since ρ^r has order m, it is also a generator of the rotation group $\mathbb{Z}_m$, and hence we may obtain L as the quotient of one B_j^{2n-1} by identifying its two faces B_j^{2n-2} and B_{j+1}^{2n-2} together via ρ^r.



In particular, when $n = 2$, B_j^{2n-1} is a lens-shaped 3-ball and L is obtained from this ball by identifying its two curved disk faces via ρ^r, which may be described as the composition of the reflection across the plane containing the rim of the lens, taking one face of the lens to the other, followed by a rotation of this face through the angle $2\pi\ell/m$ where $\ell = r\ell_1$. The figure illustrates the case $(m,\ell) = (7,2)$, with the two dots indicating a typical pair of identified points in the upper and lower faces of the lens. Since the lens space L is determined by the rotation angle $2\pi\ell/m$, it is conveniently written $L_{\ell/m}$. Clearly only the mod m value of ℓ matters. It is a classical theorem of Reidemeister from the 1930s that $L_{\ell/m}$ is homeomorphic to $L_{\ell'/m'}$ iff $m' = m$ and $\ell' \equiv \pm\ell^{\pm 1}$ mod m. For example, when $m = 7$ there are only two distinct lens spaces $L_{1/7}$ and $L_{2/7}$. The 'if' part of this theorem is easy: Reflecting the lens through a mirror shows that $L_{\ell/m} \approx L_{-\ell/m}$, and by interchanging the roles of the two $\mathbb{C}$ factors of $\mathbb{C}^2$ one obtains $L_{\ell/m} \approx L_{\ell^{-1}/m}$. In the converse direction, $L_{\ell/m} \approx L_{\ell'/m'}$ clearly implies $m = m'$ since $\pi_1(L_{\ell/m}) \approx \mathbb{Z}_m$. The rest of the theorem takes considerably more work, involving either special 3-dimensional techniques or more algebraic methods that generalize to classify the higher-dimensional lens spaces as well. The latter approach is explained in [Cohen 1973].

Returning to the construction of a CW structure on $L_m(\ell_1, \cdots, \ell_n)$, observe that the $(2n-3)$-dimensional lens space $L_m(\ell_1, \cdots, \ell_{n-1})$ sits in $L_m(\ell_1, \cdots, \ell_n)$ as the quotient of S^{2n-3}, and $L_m(\ell_1, \cdots, \ell_n)$ is obtained from this subspace by attaching two cells, of dimensions $2n-2$ and $2n-1$, coming from the interiors of B_j^{2n-1} and its two identified faces B_j^{2n-2} and B_{j+1}^{2n-2}. Inductively this gives a CW structure on $L_m(\ell_1, \cdots, \ell_n)$ with one cell e^k in each dimension $k \le 2n-1$.

The boundary maps in the associated cellular chain complex are computed as follows. The first one, d_{2n-1}, is zero since the identification of the two faces of B_j^{2n-1} is via a reflection (degree -1) across B_j^{2n-1} fixing S^{2n-3}, followed by a rotation (degree $+1$), so $d_{2n-1}(e^{2n-1}) = e^{2n-2} - e^{2n-2} = 0$. The next boundary map d_{2n-2} takes e^{2n-2} to me^{2n-3} since the attaching map for e^{2n-2} is the quotient map $S^{2n-3} \to L_m(\ell_1, \cdots, \ell_{n-1})$ and the balls B_j^{2n-3} in S^{2n-3} which project down onto e^{2n-3} are permuted cyclically by the rotation ρ of degree $+1$. Inductively, the subsequent boundary maps d_k then alternate between 0 and multiplication by m.

Also of interest are the infinite-dimensional lens spaces $L_m(\ell_1, \ell_2, \cdots) = S^\infty/\mathbb{Z}_m$ defined in the same way as in the finite-dimensional case, starting from a sequence of integers $\ell_1, \ell_2, \cdots$ relatively prime to m. The space $L_m(\ell_1, \ell_2, \cdots)$ is the union of the increasing sequence of finite-dimensional lens spaces $L_m(\ell_1, \cdots, \ell_n)$ for $n = 1, 2, \cdots$, each of which is a subcomplex of the next in the cell structure we have just constructed, so $L_m(\ell_1, \ell_2, \cdots)$ is also a CW complex. Its cellular chain complex consists of a $\mathbb{Z}$ in each dimension with boundary maps alternately 0 and m, so its reduced homology consists of a $\mathbb{Z}_m$ in each odd dimension.

In the terminology of §1.B, the infinite-dimensional lens space $L_m(\ell_1, \ell_2, \cdots)$ is an Eilenberg–MacLane space $K(\mathbb{Z}_m, 1)$ since its universal cover S^∞ is contractible, as we showed there. By Theorem 1B.8 the homotopy type of $L_m(\ell_1, \ell_2, \cdots)$ depends only on m, and not on the ℓ_i's. This is not true in the finite-dimensional case, when two lens spaces $L_m(\ell_1, \cdots, \ell_n)$ and $L_m(\ell'_1, \cdots, \ell'_n)$ have the same homotopy type iff $\ell_1 \cdots \ell_n \equiv \pm k^n \ell'_1 \cdots \ell'_n \bmod m$ for some integer k. A proof of this is outlined in Exercise 2 in §3.E and Exercise 29 in §4.2. For example, the 3-dimensional lens spaces $L_{1/5}$ and $L_{2/5}$ are not homotopy equivalent, though they have the same fundamental group and the same homology groups. On the other hand, $L_{1/7}$ and $L_{2/7}$ are homotopy equivalent but not homeomorphic.

Euler Characteristic

For a finite CW complex X, the **Euler characteristic** $\chi(X)$ is defined to be the alternating sum $\sum_n (-1)^n c_n$ where c_n is the number of n-cells of X, generalizing the familiar formula *vertices* $-$ *edges* $+$ *faces* for 2-dimensional complexes. The following result shows that $\chi(X)$ can be defined purely in terms of homology, and hence depends only on the homotopy type of X. In particular, $\chi(X)$ is independent of the choice of CW structure on X.

Theorem 2.44. $\chi(X) = \sum_n (-1)^n \operatorname{rank} H_n(X)$.

Here the **rank** of a finitely generated abelian group is the number of $\mathbb{Z}$ summands when the group is expressed as a direct sum of cyclic groups. We shall need the following fact, whose proof we leave as an exercise: If $0 \to A \to B \to C \to 0$ is a short exact sequence of finitely generated abelian groups, then $\operatorname{rank} B = \operatorname{rank} A + \operatorname{rank} C$.

Proof of 2.44: This is purely algebraic. Let

$$0 \longrightarrow C_k \xrightarrow{d_k} C_{k-1} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{d_1} C_0 \longrightarrow 0$$

be a chain complex of finitely generated abelian groups, with cycles $Z_n = \operatorname{Ker} d_n$, boundaries $B_n = \operatorname{Im} d_{n+1}$, and homology $H_n = Z_n/B_n$. Thus we have short exact sequences $0 \to Z_n \to C_n \to B_{n-1} \to 0$ and $0 \to B_n \to Z_n \to H_n \to 0$, hence

$$\begin{aligned} \operatorname{rank} C_n &= \operatorname{rank} Z_n + \operatorname{rank} B_{n-1} \\ \operatorname{rank} Z_n &= \operatorname{rank} B_n + \operatorname{rank} H_n \end{aligned}$$

Now substitute the second equation into the first, multiply the resulting equation by $(-1)^n$, and sum over n to get $\sum_n (-1)^n \operatorname{rank} C_n = \sum_n (-1)^n \operatorname{rank} H_n$. Applying this with $C_n = H_n(X^n, X^{n-1})$ then gives the theorem. $\square$

For example, the surfaces M_g and N_g have Euler characteristics $\chi(M_g) = 2 - 2g$ and $\chi(N_g) = 2 - g$. Thus all the orientable surfaces M_g are distinguished from each other by their Euler characteristics, as are the nonorientable surfaces N_g, and there are only the relations $\chi(M_g) = \chi(N_{2g})$.

Split Exact Sequences

Suppose one has a retraction $r : X \to A$, so $ri = \mathbb{1}$ where $i : A \to X$ is the inclusion. The induced map $i_* : H_n(A) \to H_n(X)$ is then injective since $r_* i_* = \mathbb{1}$. From this it follows that the boundary maps in the long exact sequence for (X, A) are zero, so the long exact sequence breaks up into short exact sequences

$$0 \longrightarrow H_n(A) \xrightarrow{\;i_*\;} H_n(X) \xrightarrow{\;j_*\;} H_n(X, A) \longrightarrow 0$$

The relation $r_* i_* = \mathbb{1}$ actually gives more information than this, by the following piece of elementary algebra:

Splitting Lemma. *For a short exact sequence* $0 \longrightarrow A \xrightarrow{\;i\;} B \xrightarrow{\;j\;} C \longrightarrow 0$ *of abelian groups the following statements are equivalent*:

(a) *There is a homomorphism* $p : B \to A$ *such that* $pi = \mathbb{1} : A \to A$.

(b) *There is a homomorphism* $s : C \to B$ *such that* $js = \mathbb{1} : C \to C$.

(c) *There is an isomorphism* $B \approx A \oplus C$ *making a commutative diagram as at the right, where the maps in the lower row are the obvious ones,* $a \mapsto (a, 0)$ *and* $(a, c) \mapsto c$.

$$\begin{tikzcd} & & B \arrow[dr, "j"] \arrow[dd, "\approx"] & & \\ 0 \arrow[r] & A \arrow[ur, "i"] \arrow[dr] & & C \arrow[r] & 0 \\ & & A \oplus C \arrow[ur] & & \end{tikzcd}$$

If these conditions are satisfied, the exact sequence is said to **split**. Note that (c) is symmetric: There is no essential difference between the roles of A and C.

Sketch of Proof: For the implication (a) $\Rightarrow$ (c) one checks that the map $B \to A \oplus C$, $b \mapsto (p(b), j(b))$, is an isomorphism with the desired properties. For (b) $\Rightarrow$ (c) one uses instead the map $A \oplus C \to B$, $(a, c) \mapsto i(a) + s(c)$. The opposite implications (c) $\Rightarrow$ (a) and (c) $\Rightarrow$ (b) are fairly obvious. If one wants to show (b) $\Rightarrow$ (a) directly, one can define $p(b) = i^{-1}(b - sj(b))$. Further details are left to the reader. $\square$

Except for the implications (b) $\Rightarrow$ (a) and (b) $\Rightarrow$ (c), the proof works equally well for nonabelian groups. In the nonabelian case, (b) is definitely weaker than (a) and (c), and short exact sequences satisfying (b) only determine B as a semidirect product of A and C. The difficulty is that $s(C)$ might not be a normal subgroup of B. In the nonabelian case one defines 'splitting' to mean that (b) is satisfied.

In both the abelian and nonabelian contexts, if C is free then every exact sequence $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ splits, since one can define $s:C\to B$ by choosing a basis $\{c_\alpha\}$ for C and letting $s(c_\alpha)$ be any element $b_\alpha \in B$ such that $j(b_\alpha) = c_\alpha$. The converse is also true: If every short exact sequence ending in C splits, then C is free. This is because for every C there is a short exact sequence $0\to A\to B\to C\to 0$ with B free — choose generators for C and let B have a basis in one-to-one correspondence with these generators, then let $B\to C$ send each basis element to the corresponding generator — so if this sequence $0\to A\to B\to C\to 0$ splits, C is isomorphic to a subgroup of a free group, hence is free.

From the Splitting Lemma and the remarks preceding it we deduce that a retraction $r:X\to A$ gives a splitting $H_n(X) \approx H_n(A)\oplus H_n(X,A)$. This can be used to show the nonexistence of such a retraction in some cases, for example in the situation of the Brouwer fixed point theorem, where a retraction $D^n\to S^{n-1}$ would give an impossible splitting $H_{n-1}(D^n) \approx H_{n-1}(S^{n-1})\oplus H_{n-1}(D^n,S^{n-1})$. For a somewhat more subtle example, consider the mapping cylinder M_f of a degree m map $f:S^n\to S^n$ with $m>1$. If M_f retracted onto the $S^n \subset M_f$ corresponding to the domain of f, we would have a split short exact sequence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_n(S^n) & \longrightarrow & H_n(M_f) & \longrightarrow & H_n(M_f,S^n) & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\;m\;} & \mathbb{Z} & \longrightarrow & \mathbb{Z}_m & \longrightarrow & 0
\end{array}$$

But this sequence does not split since $\mathbb{Z}$ is not isomorphic to $\mathbb{Z}\oplus\mathbb{Z}_m$ if $m>1$, so the retraction cannot exist. In the simplest case of the degree 2 map $S^1\to S^1$, $z\mapsto z^2$, this says that the Möbius band does not retract onto its boundary circle.

Homology of Groups

In §1.B we constructed for each group G a CW complex $K(G,1)$ having a contractible universal cover, and we showed that the homotopy type of such a space $K(G,1)$ is uniquely determined by G. The homology groups $H_n(K(G,1))$ therefore depend only on G, and are usually denoted simply $H_n(G)$. The calculations for lens spaces in Example 2.43 show that $H_n(\mathbb{Z}_m)$ is $\mathbb{Z}_m$ for odd n and 0 for even $n>0$. Since S^1 is a $K(\mathbb{Z},1)$ and the torus is a $K(\mathbb{Z}\times\mathbb{Z},1)$, we also know the homology of these two groups. More generally, the homology of finitely generated abelian groups can be computed from these examples using the Künneth formula in §3.B and the fact that a product $K(G,1)\times K(H,1)$ is a $K(G\times H,1)$.

Here is an application of the calculation of $H_n(\mathbb{Z}_m)$:

Proposition 2.45. *If a finite-dimensional CW complex X is a $K(G,1)$, then the group $G = \pi_1(X)$ must be torsionfree.*

This applies to quite a few manifolds, for example closed surfaces other than S^2 and $\mathbb{R}P^2$, and also many 3-dimensional manifolds such as complements of knots in S^3.

Proof: If G had torsion, it would have a finite cyclic subgroup $\mathbb{Z}_m$ for some $m > 1$, and the covering space of X corresponding to this subgroup of $G = \pi_1(X)$ would be a $K(\mathbb{Z}_m, 1)$. Since X is a finite-dimensional CW complex, the same would be true of its covering space $K(\mathbb{Z}_m, 1)$, and hence the homology of the $K(\mathbb{Z}_m, 1)$ would be nonzero in only finitely many dimensions. But this contradicts the fact that $H_n(\mathbb{Z}_m)$ is nonzero for infinitely many values of n. $\square$

Reflecting the richness of group theory, the homology of groups has been studied quite extensively. A good starting place for those wishing to learn more is the textbook [Brown 1982]. At a more advanced level the books [Adem & Milgram 1994] and [Benson 1992] treat the subject from a mostly topological viewpoint.

Mayer–Vietoris Sequences

In addition to the long exact sequence of homology groups for a pair (X, A), there is another sort of long exact sequence, known as a **Mayer–Vietoris sequence**, which is equally powerful but is sometimes more convenient to use. For a pair of subspaces A, $B \subset X$ such that X is the union of the interiors of A and B, this exact sequence has the form

$$\cdots \longrightarrow H_n(A \cap B) \xrightarrow{\Phi} H_n(A) \oplus H_n(B) \xrightarrow{\Psi} H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \longrightarrow \cdots$$
$$\cdots \longrightarrow H_0(X) \longrightarrow 0$$

In addition to its usefulness for calculations, the Mayer–Vietoris sequence is also applied frequently in induction arguments, where one might know that a certain statement is true for A, B, and $A \cap B$ by induction and then deduce that it is true for $A \cup B$ by the exact sequence.

The Mayer–Vietoris sequence is easy to derive from the machinery of §2.1. Let $C_n(A+B)$ be the subgroup of $C_n(X)$ consisting of chains that are sums of chains in A and chains in B. The usual boundary map $\partial : C_n(X) \to C_{n-1}(X)$ takes $C_n(A+B)$ to $C_{n-1}(A+B)$, so the $C_n(A+B)$'s form a chain complex. According to Proposition 2.21, the inclusions $C_n(A+B) \hookrightarrow C_n(X)$ induce isomorphisms on homology groups. The Mayer–Vietoris sequence is then the long exact sequence of homology groups associated to the short exact sequence of chain complexes formed by the short exact sequences

$$0 \longrightarrow C_n(A \cap B) \xrightarrow{\varphi} C_n(A) \oplus C_n(B) \xrightarrow{\psi} C_n(A+B) \longrightarrow 0$$

where $\varphi(x) = (x, -x)$ and $\psi(x, y) = x + y$. The exactness of this short exact sequence can be checked as follows. First, $\operatorname{Ker}\varphi = 0$ since a chain in $A \cap B$ that is zero as a chain in A (or in B) must be the zero chain. Next, $\operatorname{Im}\varphi \subset \operatorname{Ker}\psi$ since $\psi\varphi = 0$. Also, $\operatorname{Ker}\psi \subset \operatorname{Im}\varphi$ since for a pair $(x, y) \in C_n(A) \oplus C_n(B)$ the condition $x + y = 0$ implies $x = -y$, so x is a chain in both A and B, that is, $x \in C_n(A \cap B)$, and $(x, y) = (x, -x) \in \operatorname{Im}\varphi$. Finally, exactness at $C_n(A + B)$ is immediate from the definition of $C_n(A + B)$.

The boundary map $\partial : H_n(X) \to H_{n-1}(A \cap B)$ can easily be made explicit. A class $\alpha \in H_n(X)$ is represented by a cycle z, and by barycentric subdivision or some other method we can choose z to be a sum $x + y$ of chains in A and B, respectively. It need not be true that x and y are cycles individually, but $\partial x = -\partial y$ since $\partial(x + y) = 0$, and the element $\partial\alpha \in H_{n-1}(A \cap B)$ is represented by the cycle $\partial x = -\partial y$, as is clear from the definition of the boundary map in the long exact sequence of homology groups associated to a short exact sequence of chain complexes.

There is also a formally identical Mayer–Vietoris sequence for reduced homology groups, obtained by augmenting the previous short exact sequence of chain complexes in the obvious way:

$$\begin{tikzcd}
0 \arrow[r] & C_0(A \cap B) \arrow[r, "\varphi"] \arrow[d, "\varepsilon"] & C_0(A) \oplus C_0(B) \arrow[r, "\psi"] \arrow[d, "\varepsilon \oplus \varepsilon"] & C_0(A + B) \arrow[r] \arrow[d, "\varepsilon"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\varphi"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "\psi"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

Mayer–Vietoris sequences can be viewed as analogs of the van Kampen theorem since if $A \cap B$ is path-connected, the H_1 terms of the reduced Mayer–Vietoris sequence yield an isomorphism $H_1(X) \approx \big(H_1(A) \oplus H_1(B)\big) / \operatorname{Im}\Phi$. This is exactly the abelianized statement of the van Kampen theorem, and H_1 is the abelianization of π_1 for path-connected spaces, as we show in §2.A.

There are also Mayer–Vietoris sequences for decompositions $X = A \cup B$ such that A and B are deformation retracts of neighborhoods U and V with $U \cap V$ deformation retracting onto $A \cap B$. Under these assumptions the five-lemma implies that the maps $C_n(A + B) \to C_n(U + V)$ induce isomorphisms on homology, and hence so do the maps $C_n(A + B) \to C_n(X)$, which was all that we needed to obtain a Mayer–Vietoris sequence. For example, if X is a CW complex and A and B are subcomplexes, then we can choose for U and V neighborhoods of the form $N_\varepsilon(A)$ and $N_\varepsilon(B)$ constructed in the Appendix, which have the property that $N_\varepsilon(A) \cap N_\varepsilon(B) = N_\varepsilon(A \cap B)$.

Example 2.46. Take $X = S^n$ with A and B the northern and southern hemispheres, so that $A \cap B = S^{n-1}$. Then in the reduced Mayer–Vietoris sequence the terms $\widetilde{H}_i(A) \oplus \widetilde{H}_i(B)$ are zero, so we obtain isomorphisms $\widetilde{H}_i(S^n) \approx \widetilde{H}_{i-1}(S^{n-1})$. This gives another way of calculating the homology groups of S^n by induction.

Example 2.47. We can decompose the Klein bottle K as the union of two Möbius bands A and B glued together by a homeomorphism between their boundary circles.

Then A, B, and $A\cap B$ are homotopy equivalent to circles, so the interesting part of the reduced Mayer–Vietoris sequence for the decomposition $K = A\cup B$ is the segment

$$0 \longrightarrow H_2(K) \longrightarrow H_1(A\cap B) \xrightarrow{\Phi} H_1(A)\oplus H_1(B) \longrightarrow H_1(K) \longrightarrow 0$$

The map Φ is $\mathbb{Z}\to\mathbb{Z}\oplus\mathbb{Z}$, $1\mapsto(2,-2)$, since the boundary circle of a Möbius band wraps twice around the core circle. Since Φ is injective we obtain $H_2(K)=0$. Furthermore, we have $H_1(K)\approx\mathbb{Z}\oplus\mathbb{Z}_2$ since we can choose $(1,0)$ and $(1,-1)$ as a basis for $\mathbb{Z}\oplus\mathbb{Z}$. All the higher homology groups of K are zero from the earlier part of the Mayer–Vietoris sequence.

Example 2.48. Let us describe an exact sequence which is somewhat similar to the Mayer–Vietoris sequence and which in some cases generalizes it. If we are given two maps $f,g:X\to Y$ then we can form a quotient space Z of the disjoint union of $X\times I$ and Y via the identifications $(x,0)\sim f(x)$ and $(x,1)\sim g(x)$, thus attaching one end of $X\times I$ to Y by f and the other end by g. For example, if f and g are each the identity map $X\to X$ then $Z = X\times S^1$. If only one of f and g, say f, is the identity map, then Z is homeomorphic to what is called the mapping torus of g, the quotient space of $X\times I$ under the identifications $(x,0)\sim(g(x),1)$. The Klein bottle is an example, with g a reflection $S^1\to S^1$.

The exact sequence we want has the form

$$(*)\qquad \cdots\longrightarrow H_n(X)\xrightarrow{f_*-g_*}H_n(Y)\xrightarrow{i_*}H_n(Z)\longrightarrow H_{n-1}(X)\xrightarrow{f_*-g_*}H_{n-1}(Y)\longrightarrow\cdots$$

where i is the evident inclusion $Y\hookrightarrow Z$. To derive this exact sequence, consider the map $q:(X\times I, X\times\partial I)\to(Z,Y)$ that is the restriction to $X\times I$ of the quotient map $X\times I\amalg Y\to Z$. The map q induces a map of long exact sequences:

$$\begin{tikzcd}
\cdots \arrow[r, "0"] & H_{n+1}(X\times I, X\times\partial I) \arrow[r, "\partial"] \arrow[d, "q_*"] & H_n(X\times\partial I) \arrow[r, "i_*"] \arrow[d, "q_*"] & H_n(X\times I) \arrow[r, "0"] \arrow[d, "q_*"] & \cdots \\
\cdots \arrow[r] & H_{n+1}(Z,Y) \arrow[r, "\partial"] & H_n(Y) \arrow[r, "i_*"] & H_n(Z) \arrow[r] & \cdots
\end{tikzcd}$$

In the upper row the middle term is the direct sum of two copies of $H_n(X)$, and the map i_* is surjective since $X\times I$ deformation retracts onto $X\times\{0\}$ and $X\times\{1\}$. Surjectivity of the maps i_* in the upper row implies that the next maps are 0, which in turn implies that the maps ∂ are injective. Thus the map ∂ in the upper row gives an isomorphism of $H_{n+1}(X\times I, X\times\partial I)$ onto the kernel of i_*, which consists of the pairs $(\alpha,-\alpha)$ for $\alpha\in H_n(X)$. This kernel is a copy of $H_n(X)$, and the middle vertical map q_* takes $(\alpha,-\alpha)$ to $f_*(\alpha)-g_*(\alpha)$. The left-hand q_* is an isomorphism since these are good pairs and q induces a homeomorphism of quotient spaces $(X\times I)/(X\times\partial I)\to Z/Y$. Hence if we replace $H_{n+1}(Z,Y)$ in the lower exact sequence by the isomorphic group $H_n(X)\approx\operatorname{Ker} i_*$ we obtain the long exact sequence we want.

In the case of the mapping torus of a reflection $g:S^1\to S^1$, with Z a Klein bottle, the interesting portion of the exact sequence $(*)$ is

$$\begin{array}{ccccccccccccc} 0 \longrightarrow H_2(Z) \longrightarrow & H_1(S^1) & \xrightarrow{\mathbb{1}-g_*} & H_1(S^1) & \longrightarrow H_1(Z) \longrightarrow & H_0(S^1) & \xrightarrow{\mathbb{1}-g_*} & H_0(S^1) \\ & \| & & \| & & \| & & \| \\ & \mathbb{Z} & \xrightarrow{\quad 2 \quad} & \mathbb{Z} & & \mathbb{Z} & \xrightarrow{\quad 0 \quad} & \mathbb{Z} \end{array}$$

Thus $H_2(Z) = 0$ and we have a short exact sequence $0 \to \mathbb{Z}_2 \to H_1(Z) \to \mathbb{Z} \to 0$. This splits since $\mathbb{Z}$ is free, so $H_1(Z) \approx \mathbb{Z}_2 \oplus \mathbb{Z}$. Other examples are given in the Exercises.

If Y is the disjoint union of spaces Y_1 and Y_2, with $f : X \to Y_1$ and $g : X \to Y_2$, then Z consists of the mapping cylinders of these two maps with their domain ends identified. For example, suppose we have a CW complex decomposed as the union of two subcomplexes A and B and we take f and g to be the inclusions $A \cap B \hookrightarrow A$ and $A \cap B \hookrightarrow B$. Then the double mapping cylinder Z is homotopy equivalent to $A \cup B$ since we can view Z as $(A \cap B) \times I$ with A and B attached at the two ends, and then slide the attaching of A down to the B end to produce $A \cup B$ with $(A \cap B) \times I$ attached at one of its ends. By Proposition 0.18 the sliding operation preserves homotopy type, so we obtain a homotopy equivalence $Z \simeq A \cup B$. The exact sequence $(*)$ in this case is the Mayer–Vietoris sequence.

A relative form of the Mayer–Vietoris sequence is sometimes useful. If one has a pair of spaces $(X, Y) = (A \cup B, C \cup D)$ with $C \subset A$ and $D \subset B$, such that X is the union of the interiors of A and B, and Y is the union of the interiors of C and D in Y, then there is a relative Mayer–Vietoris sequence

$$\cdots \longrightarrow H_n(A \cap B, C \cap D) \xrightarrow{\Phi} H_n(A, C) \oplus H_n(B, D) \xrightarrow{\Psi} H_n(X, Y) \xrightarrow{\partial} \cdots$$

To derive this, consider the commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_n(C \cap D) & \xrightarrow{\varphi} & C_n(C) \oplus C_n(D) & \xrightarrow{\psi} & C_n(C + D) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_n(A \cap B) & \xrightarrow{\varphi} & C_n(A) \oplus C_n(B) & \xrightarrow{\psi} & C_n(A + B) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_n(A \cap B, C \cap D) & \xrightarrow{\varphi} & C_n(A, C) \oplus C_n(B, D) & \xrightarrow{\psi} & C_n(A + B, C + D) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

where $C_n(A + B, C + D)$ is the quotient of the subgroup $C_n(A + B) \subset C_n(X)$ by its subgroup $C_n(C + D) \subset C_n(Y)$. Thus the columns of the diagram are exact. We have seen that the first two rows are exact, and we claim that the third row is exact also, with the maps φ and ψ induced from the φ and ψ in the second row. Since $\psi\varphi = 0$ in the second row, this holds also in the third row, so the third row is at least a chain complex. Viewing the three rows as chain complexes, the diagram then represents a short exact sequence of chain complexes. The associated long exact sequence of homology groups has two out of every three terms zero since the first two rows of the diagram are exact. Hence the remaining homology groups are zero and the third row is exact.