CW complexes

Def A CW cplx is constructed as follows.

1). X^0 : discrete points

2) inductively build X^{n+1} from X^n

$$\begin{tikzcd} J \times S^n \arrow[r, "(\varphi_j)_{j\in J}"] \arrow[d, hook] & X^n \arrow[d] \\ J \times D^{n+1} \arrow[r] & X^{n+1} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

3) $\operatorname{colim}(X^0 \to X^1 \to X^2 \to \cdots) = X$

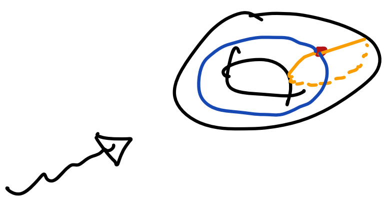
X^n is called the n-th skeleton.

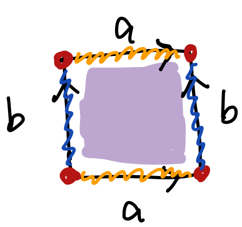
Ex: 1) graphs X^0 : vertice

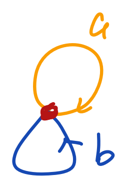
glue J copies of edges.

$$\begin{tikzcd} J \times S^0 \arrow[r] \arrow[d, hook] & X^0 \arrow[d] \\ J \times D^1 \arrow[r] & X^1 \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$ $\leftarrow$ a graph

2) torus.



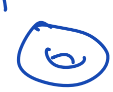


glue  $= D^1$ to 

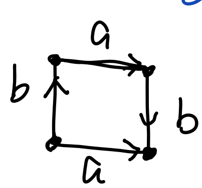
$= X^1$

via

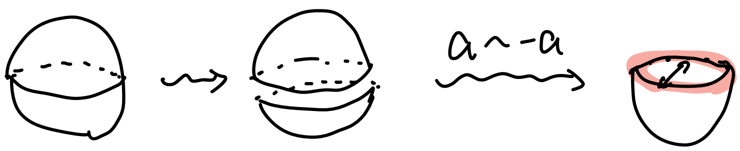
$$\begin{tikzcd} S^1 \arrow[r, "ab^{-1}a^{-1}b"] \arrow[d] & X^1 \arrow[d] \\ D^1 \arrow[r] & \end{tikzcd}$$



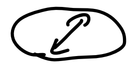
3) Klein bottle

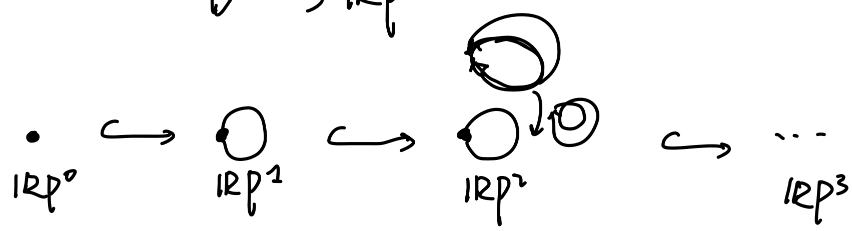


4) $\mathbb{R}P^n$ $\qquad$ $S^n \xrightarrow[\substack{\text{gluing} \\ a\sim -a}]{} \mathbb{R}P^n$

$\mathbb{R}P^2$: $S^2 =$  $\rightsquigarrow \mathbb{R}P^2$

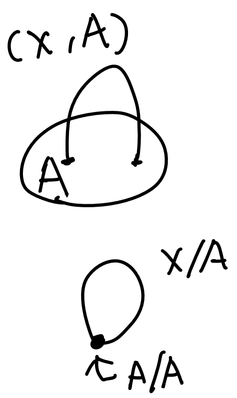
$$\begin{tikzcd} S^1 \arrow[r, "a\sim -a"] \arrow[d] & \mathbb{R}P^1 \arrow[d] \\ D^2 \arrow[r] & \mathbb{R}P^2 \end{tikzcd} \qquad \begin{tikzcd} S^{n-1} \arrow[r, "a\sim -a"] \arrow[d] & \mathbb{R}P^{n-1} \arrow[d] \\ D^n \arrow[r] & \mathbb{R}P^n \end{tikzcd}$$

 $= S^1/a\sim a$
$= \mathbb{R}P^1$



<u>Def</u>. (X,A) is a relative CW complex
if X is constructed from A by
replacing X^0 with A in CW cplx construction.

<u>Prop</u> 1) (quotient)



(X,A) is a relative CW complex,
then X/A is a CW complex s.t.

$n\geq 1$ $\quad$ n cells in (X,A) $\leftrightsquigarrow$ n cells in X/A.

2) X, Y both CW cplx.
base pts $\in X^0$ (Y^0)
$\Rightarrow X \vee Y$ is a CW cplx

n cells $(X\vee Y) = n$ cells $(X) \sqcup n$ cells (Y)
$(n\geq 1)$

terminology: $f: X \to Y$ between CW cplx.
f is <u>cellular</u> if $f(X^n) \subset Y^n$.

3) $f: A \to Y$ cellular map. X, Y both CW cplx
(pushout) A is a subcomplex of X

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook] & Y \arrow[d] \\ X \arrow[r] & Y \coprod_f X \end{tikzcd}$$

$Y \coprod_f X$ is also a CW cplx.

4). X, Y CW cplx
$\Rightarrow X \times Y$ is also a CW cplx.

$$n\text{-cell}(X \times Y) = \coprod_{0 \le j \le n} j\text{-cells}(X) \times n\text{-}j \text{ cells}(Y)$$

$$D^j \times D^{n-j} = D^n$$

In particular $X \times I$ is a CW cplx.

<u>Last time</u> $\pi_n(X)$ higher homotopy groups.

$f: X \xrightarrow{\simeq} Y \Rightarrow \pi_n(f): \pi_n(X) \xrightarrow{\cong} \pi_n(Y)$

Q: whether $\Leftarrow$ is true?

A: not in general.

When X, Y CW, $\Leftarrow$ true ✓.

① says that CW complexes are good.

② "CW Approximation" says that considering CW complexes is not very far from considering all topological spaces

① Whitehead theorem

② CW approximation.

Def. (n-equivalence) $f: A \to X$ is an n-equivalence if $\pi_q(f)$ is surj for $q \le n$. inj for $q < n$

$n = \infty \Rightarrow \pi_n(A) \xrightarrow{\pi_n f} \pi_n(X)$. iso for all n.

① Th'm (Whitehead).

X CW cplx.

$f: Y \to Z$ an n-equivalence.

Then $f_*: [X, Y] \to [X, Z]$ is a bijection if $\dim X < n$, surj if $\dim X = n$.

Take $n = \infty$, $X = Z$, Y CW cplx.

Whitehead $\leadsto$ $f: Y \to Z$ induces iso on π_n for all n.

$f_*: [Z, Y] \to [Z, Z]$ is bijection

$g \longmapsto \mathrm{id}$

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[rr, bend right, "\mathrm{id}"'] & Y \arrow[r, "f"] & Z \end{tikzcd}$$

$\leadsto$ f is a h.e.

② (Th'm) X, Y CW cplx.

$f: X \to Y$. Then $f \simeq f'$ & f' is cellular.

(Th'm) X any space.

$\exists$ PX a CW cplx & $r: PX \to X$

$\pi_*(-)$ is iso

weak equivalence

such that for $f: X \to Y$, $\exists$ $Pf: PX \to PY$

such that
$$\begin{tikzcd} PX \arrow[r, "Pf"] \arrow[d, "r_X"'] & PY \arrow[d, "r_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$
commutes up to homotopy.

"functoriality" statement

$\pi_*(f)$ is iso $\Rightarrow$ $\exists$ inverse of $\pi_*(f)$ on algebra level

$g': \pi_*(Y) \to \pi_*(X)$ s.t

$g \circ \pi_*(f) = id$

$\pi_*(f) \circ g = id$.

g' not nec. come from $\pi_*(g)$ for some $g: Y \to X$ on topological level

Why "weak" equivalence

Story.

Study things up to homotopy

$\Rightarrow$ $\pi_*(-)$ this has strong control

not full control

over homotopy type

$\rightsquigarrow$ study things up to weak equivalences

$W = \{\text{weak equivalences}\} \subset \mathrm{Morph}(\mathrm{Top})$

$\mathrm{Top}[W^{-1}]$ $\leftarrow$ add inverses to things in W.

$X \xrightarrow{w.e.} Y \overset{\text{add}}{\rightsquigarrow} X \leftarrow Y$

$X \xleftarrow{w.e.} X_1 \xleftarrow{w.e.} X_2 \to X_3 \to X_4 \xleftarrow{w.e} Z \quad \in \mathrm{Morph}(\mathrm{Top}[W^{-1}])$

$\underbrace{\mathrm{Top}[W^{-1}]}_{\text{huge}} \simeq \text{CW cplx.}$ $\leftarrow$ smaller

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② CW approximation

Recall

$\pi_1(S^1) = \mathbb{Z}$

$\mathbb{Z} \to \mathbb{R} \to S^1$ $\quad \pi_{\geq 2}(S^1) = 0$

$\pi_n(\mathbb{Z}) \to \pi_n(\mathbb{R}) \to \pi_n(S^1) \to \pi_{n-1}(\mathbb{Z}) \to$

$k \geq 1 \quad \pi_n(S^{n+k}) = 0$

$S^1 \to S^3 \to S^2$ $\quad \pi_1(S^2) = 0$, $\quad \pi_2(S^2) = \mathbb{Z}$

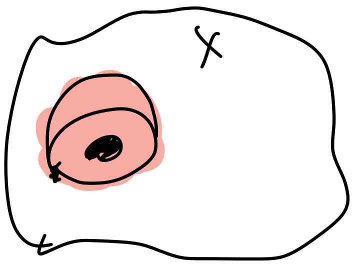
X : topological space, connected

$\pi_1(X)$ = generators / relations
$\alpha_1, \alpha_2, \dots$

The map $f_i : \bigvee_{\alpha_i} S^1 \xrightarrow{\vee \alpha_i} X$ is surjective on π_1

① $(PX)_1 = \bigvee_{\alpha_i} S^1$

② 1): attaching 2-cells to $(PX)_1$ to create relations



$$\begin{tikzcd} S^1 \arrow[r, "f"] \arrow[d, hook] & (PX)_1 \arrow[d, "i"] \\ D^2 \arrow[r] & Y \end{tikzcd}$$

defines $\pi_1((PX)_1) = \langle a, b \rangle$

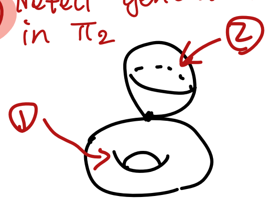
$(i \circ f : S^1 \to Y) \sim 0$

$\rightsquigarrow \pi_1(Y) = \langle a, b \rangle / f \sim 0$

obtain PX_2'

f_1 extends to $f_1' : PX_2' \to X$ s.t. $\pi_1(f_1')$ is iso.

2-cells are used for
1) create relations in π_1
2) detect generators in π_2



2) attach 2-cells to approximate $\pi_2(X)$.

$\pi_2(X_2)$ = gen / relations.

For each generator, wedge a S^2

$(PX)_2 = (PX)_2' \vee (\bigvee_{\alpha_i} S^2)$ $\rightsquigarrow$ the map $(PX)_2 \xrightarrow{f_1' \vee \alpha_i^2} X$ induces iso on π_2.

③ attach 3-cells, this doesn't change π_1

$$\operatorname{colim}\left((PX)_1 \to (PX)_2 \to (PX)_3 \to \cdots\right) = PX$$

$$\pi_n((PX)_{n+k}) = \pi_n(X) \qquad k \geq 1$$

$\rightsquigarrow k \to \infty$: $\pi_n(PX) \xrightarrow{\cong} \pi_n(X)$ $\forall n$.

this map is constructed along the way.

① Th'm (Whitehead).

X CW cplx.

$f: Y \to Z$ an n-equivalence.

Then $f_*: [X, Y] \to [X, Z]$ is a bijection if $\underline{\dim X} < n$, surj if $\dim X = n$.

→ largest deg of cells in X

Pf sketch

bijection:

surj:
$$\begin{tikzcd} & Y \arrow[d] \\ X \arrow[r] \arrow[ur, dashed] & Z \end{tikzcd}$$

consider $X = S^{n-k}$

use n-equivalence ✓

inj:
$$\begin{tikzcd} X \times : \arrow[r] \arrow[d] & Y \arrow[d] \\ X \times I \arrow[r] \arrow[ur, dashed] & Z \end{tikzcd}$$

+ show property preserved under

$$\operatorname{colim}\left(\begin{tikzcd} \circ \arrow[r] \arrow[d] & \circ \\ \circ & \end{tikzcd} \right)$$

Th'm (Freudenthal)

Consider the map $\Sigma : \pi_n(X) \to \pi_{n+1}(\Sigma X)$.

$$[S^n, X] \to [S^n \wedge S^1, X \wedge S^1] = [S^{n+1}, \Sigma X]$$

If X is m-connected, then

Σ is bijection $n < 2m+1$

surjection $n = 2m+1$

For a fixed X. the sequence

$$\to \pi_n X \to \pi_{n+1}\Sigma X \to \pi_{n+2}\Sigma^2 X \to \cdots$$

stablize.

stable homotopy group

$$\pi_n^{st}(X) = \operatorname*{colim}_{k\to\infty}(\pi_{n+k}(\Sigma^k X))$$

Chain complex

Def. $\cdots \to X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} X_{n-2} \to \cdots$

Each X_n is an abelian group
($\mathbb{Z}$-module)

Differentials d_n $\mathbb{Z}$module maps.

s.t. $d_n \circ d_{n+1} = 0$.

Def. Homology of a chain complex $\{X_n\}$

$$H_n(\{X_n\}) = \ker d_n / \operatorname{Im} d_{n+1}$$

$$X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1}$$

$d_n \circ d_{n+1} = 0 \Rightarrow \operatorname{Im} d_{n+1} \subseteq \ker d_n$

Def. short exact sequence:

$$0 \to X'' \to X \to X' \to 0 \quad \text{s.t.}$$

exact at all places.

Ex Short exact seq for chain cplx.

$$\begin{tikzcd}
 & X''_{n+1} \arrow[r] \arrow[d] & X_{n+1} \arrow[r] \arrow[d] & X'_{n+1} \arrow[r] \arrow[d] & \\
0 \arrow[ur] \arrow[r] \arrow[dr] & X''_n \arrow[r] \arrow[d] & X_n \arrow[r] \arrow[d] & X'_n \arrow[r] \arrow[d] & 0 \\
 & X''_{n-1} \arrow[r] & X_{n-1} \arrow[r] & X'_{n-1} \arrow[ur] &
\end{tikzcd}$$

$b \longmapsto a$

$c \longmapsto \bar{b} \longmapsto 0$

exact at each level.

Thm. long exact sequence for $H_\bullet(-)$

$$H_n(X''_\bullet) \to H_n(X_\bullet) \to H_n(X'_\bullet) \xrightarrow{\partial} H_{n-1}(X''_\bullet)$$

∂ connecting homomorphism

$a \longmapsto c$

check well definedness