(X, A) pairs

$\pi_n(X, A) = $ all maps $(D^n, S^{n-1}) \to (X, A)$ / homotopy equivalence.

$D^n / S^{n-1} \simeq S^n$

when $A = *$ $\rightsquigarrow$ $\pi_n(X, *)$ usual definition

LES pair (X, A) $i: A \hookrightarrow X$

$$\to \pi_n(A) \xrightarrow{\pi_n(i)} \pi_n(X) \to \pi_n(X, A) \xrightarrow[\partial]{\text{boundary}} \pi_{n-1}(A) \to$$

$\pi_n(X) = \pi_n(X, *)$

induces

$(X, *) \to (X, A)$

$(D^n, S^{n-1}) \to (X, A)$

LES associated to fiber sequences $F \to E \to B$.

$A \to F$, $X \to E$

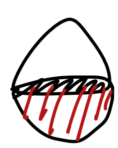
$\pi_n(E, F) = \pi_n(B)$.

---

suspension th'm.

$\pi_n(X) \xrightarrow{\Sigma} \pi_{n+1}(\Sigma X)$ is iso for n large enough.

$\Sigma X = $



CX^+

X

CX^-

$(\Sigma X, CX^-) \rightsquigarrow \pi_n(CX) \xrightarrow{=0} \pi_n(\Sigma X) \xrightarrow{\cong} \pi_n(\Sigma X, CX) \to \pi_{n-1}(CX) {=0}$

$(CX^+, X) \rightsquigarrow \pi_n(X) \to \pi_n(\cancel{X}CX) \to \pi_n(CX, X) \xrightarrow{\cong} \pi_{n-1}(X)$

$\to \pi_{n-1}(\cancel{X}CX) \to$

- homotopy excision thm only holds under some dimension assumptions.

Homology

Hatcher §2

1) simplicial homlgy
2) singular homlgy
3) Δ equivalent to 2)

May.

1) axiom for homolgy
2) cellular homlgy H^{cell}

Any homology theory satisfy axioms $\longrightarrow H \xrightarrow{\cong} H^{cell}$

3) property from the axioms.

Homology theory

H_*: pairs of spaces $\longrightarrow$ graded Abelian groups.

$+ \ \partial$ connecting homomorphism.

notation: $H_n(X, \phi) = H_n(X)$

$$H_n(X, A) \xrightarrow{\partial} H_{n-1}(A, \phi) = H_{n-1}(A)$$

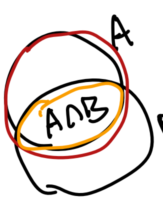
satisfying:

①. (exactness)

$$\cdots \longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \longrightarrow \cdots$$

② (excision) $(X; A, B)$ excisive triad $A^\circ \cup B^\circ = X$

$$H_*(A, A\cap B) \xrightarrow{\cong} H_*(X, B)$$



③. $(X, A) \longrightarrow (Y, B)$ weak equivalence (π_* isomorphism)

(weak equivalence) then $H_*(X, A) \xrightarrow{\cong} H_*(Y, B)$

④. (additivity) $H_*(\amalg X_i, \amalg A_i) = \bigoplus H_*(X_i, A_i)$

⑤ (dimension) $H_*(pt) = \begin{cases} \pi & * = 0 \\ 0 & * > 0. \end{cases}$ π is a Abelian grp.

$\pi = \mathbb{Z}$

<u>Def</u>. reduced homology

$$\tilde{H}_*(X) := H_*(X, *)$$

use. LES for $(X, *)$

$$\to H_*(*) \to H_*(X) \to \underset{\displaystyle \tilde{H}_*(X)}{\overset{\shortparallel}{H_*(X,*)}} \to H_{*-1}(*) \to$$

$$\rightsquigarrow \quad H_*(X) = \begin{cases} \tilde{H}_*(X) & * \geq 1 \\ \tilde{H}_*(X) \oplus \mathbb{Z} & * = 0 \end{cases}$$

$$\rightsquigarrow 1 \quad \tilde{H}_*(S^0) = H_*(pt) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & * > 0 \end{cases}$$

<u>excision axiom</u> + <u>LES</u> $\rightsquigarrow$ 2 $\tilde{H}_*(X) \cong \tilde{H}_{*+1}(\Sigma X)$ (similar in π_* case)

$\Sigma X =$ 

$$① + ② \rightsquigarrow \tilde{H}_*(S^n) = \begin{cases} \mathbb{Z} & * = n \\ 0 & \text{otherwise.} \end{cases}$$

<u>cellular homology</u>

idea: construct a chain cplx $C_*(X)$ from a cw cplx $X \rightsquigarrow H_*(C_*(X))$

1) $C_n(X)$: $X^n / X^{n-1} = \vee S^n$

$$\begin{array}{ccc} \vee S^{n-1} & \longrightarrow & X^{n-1} \\ \downarrow & & \downarrow \\ \vee D^n & \longrightarrow & X^n \end{array}$$

$C_n(X) = \bigoplus \mathbb{Z}$ a $\mathbb{Z}$ for each cell

$= \pi_n(X^n / X^{n-1})$

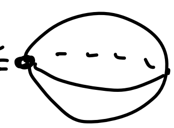
2) differentials :

$$\begin{array}{ccc} d_n : C_n(X) & \longrightarrow & C_{n-1}(X) \\ \| & & \| \\ \pi_n(X^n/X^{n-1}) & & \pi_{n-1}(X^{n-1}/X^{n-2}) \\ \| & \nearrow \partial & \\ \pi_n(X^n/X^{n-2}, X^{n-1}/X^{n-2}) & & \end{array}$$

$\partial \longrightarrow$ bdry map for $(X^n/X^{n-2}, X^{n-1}/X^{n-2})$

check: $d_n \circ d_{n+1} = 0$.

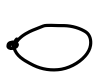
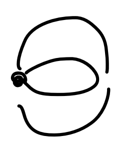
Ex : $X =$  S^2

1) $X^0 = \bullet$ $X^1 = \bullet$ $X^2 =$ 

$$\mathbb{Z} \xleftarrow{0} 0 \xleftarrow{0} \mathbb{Z}$$

$\Rightarrow H_*(S^2) = \begin{cases} \mathbb{Z} & 0, 2 \\ 0 & \text{o.w.} \end{cases}$

$\left(\tilde{H}_*(S^2) = \begin{cases} \mathbb{Z} & 2 \\ 0 & \text{o.w.} \end{cases} \right)$

2) $X^0 = \bullet$ $X^1 =$  $X^2 =$ 

$$0 \longleftarrow \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{(1,1)} \mathbb{Z} \oplus \mathbb{Z} \longleftarrow 0 \longleftarrow$$

ker/Im = $\mathbb{Z}$

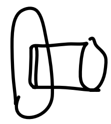
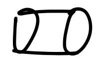
ker/Im = $\mathbb{Z}/\mathbb{Z} = 0$

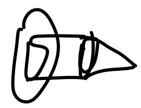
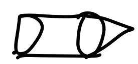
ker/Im = $\dfrac{\langle (1,-1) \rangle \cong \mathbb{Z}}{0} = \mathbb{Z}$

Oct 30 Properties deduced from axioms

① $A \to X$ is a cofibration, then

$$\begin{array}{ccc} H_*(X,A) & \xrightarrow{\cong} & \tilde{H}_*(X/A) \\ & \searrow^{\text{quotient}} & \parallel \\ & & H_*(X/A, *) \end{array}$$

Pf. $H_*(X, A) \overset{\text{w.e.}}{\cong} H_*($  ,  $)$

$\overset{\text{excision}}{\cong} H_*($  ,  $)$

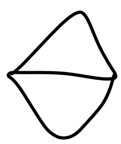
$\overset{\text{w.e.}}{\cong} H_*(X/A, *)$

$(Ci, A) \simeq (X/A, *)$

$\overset{\text{def}}{\cong} \tilde{H}_*(X/A)$

② (suspension for reduced homology)

$$\tilde{H}_*(X) \cong \tilde{H}_{*+1}(\Sigma X)$$

Pf. $\Sigma X =$  CX^+, CX^-

Let $A = CX^+$, then $A \wedge B = X$.
$B = CX^-$

same proof as in the Freudenthal suspension thm. (see last lecture).

③ Mayer-Vietoris sequence

$(X; A, B)$ triad $\quad C = A \cap B$, i.e.

$$\begin{tikzcd} C \arrow[r, "i^A"] \arrow[d, "i^B"'] & A \arrow[d, "i_A"] \\ B \arrow[r, "i_B"'] & X \end{tikzcd}$$

$\exists$ exact sequence.

minus sign

$$\to H_*(C) \xrightarrow{(i^A_*,\, i^B_*)} H_*(A) \oplus H_*(B) \xrightarrow{\begin{pmatrix} i_{A*} \\ -i_{B*} \end{pmatrix}} H_*(X) \xrightarrow{(*)} H_{*-1}(C) \to$$

$(*)$ is $\quad H_*(X) \longrightarrow H_*(X, B)$

(excision) $\cong$

$$H_*(A, C) \xrightarrow{\partial} H_{*-1}(C)$$

$$\begin{tikzcd} C \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & X \end{tikzcd}$$

Intuitively

" $X = A + B - C$ "

Pf: By exactness & excision, we have a map between two long exact seq.

$$\begin{tikzcd} \to H_{*+1}(A,C) \arrow[r] \arrow[d] & H_*(C) \arrow[r] \arrow[d] & H_*(A) \arrow[r] \arrow[d] & H_*(A,C) \arrow[r] \arrow[d, "\cong"] & H_{*-1}(C) \to \arrow[d] \\ \to H_{*+1}(X,B) \arrow[r] & H_*(B) \arrow[r] & H_*(X) \arrow[r] & H_*(X,B) \arrow[r] & H_{*-1}(B) \to \end{tikzcd}$$

Show exactness:

① at $H_*(A) \oplus H_*(B)$

want to show $\ker = \operatorname{Im}$

i.e. $a \in H_*(A)$, $b \in H_*(B)$ s.t. $i_{A*}(a) = i_{B*}(b)$

$\Leftrightarrow$ $\exists$ $c \in H_*(C)$ s.t.

$i^A_*(c) = a$ $\quad$ $i^B_*(c) = b$.

$\Leftarrow$: By commutativity.

$$\begin{array}{ccc} c & \longmapsto & a \\ \downarrow & & \downarrow \\ b & \longmapsto & i_{A*}(a) = i_{B*}(b) \end{array}$$

$\Rightarrow$: 1) a lifts. Choose a lift c'

not nec. unique

$$\begin{array}{ccccc} c' & \mapsto & a & \longmapsto & 0 \\ & & \downarrow & & \downarrow \\ & & i_{B*}(b) = i_{A*}(a) & \longmapsto & 0 \end{array}$$

2) $$\begin{array}{c} c' \\ \downarrow \\ b' \end{array}$$

b' not nec. $= b$, need to add a fix term.

i.e. c'' s.t. $$\begin{array}{c} c'' \\ \downarrow \\ b - b' \end{array}$$

consider $b - b'$

$b - b' \longmapsto 0 \iff b - b'$ lifts to $H_{*+1}(X, B)$

$$\begin{array}{ccc} d' & \longmapsto & c'' \\ \downarrow & & \downarrow \\ d & \longmapsto & b-b' \end{array} \quad \leftarrow \text{ found } c'', \text{ s.t.}$$

① $(c''+c')$
$$\downarrow$$
$$(b-b')+b' = b$$

② $(c''+c') \longmapsto 0+a=a.$

✓.

② cut $H_*(X)$ } exercise.

③ cut $H_*(C)$ }

---

4). $X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots$

$$H_*(\underbrace{\mathrm{colim}}_{\text{colim in spaces}} X_i) = \underbrace{\mathrm{colim}}_{\text{colim in Ab grps.}} H_*(X_i)$$

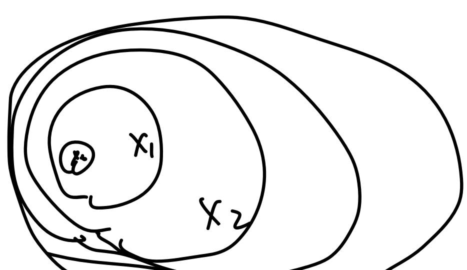
---

aside

for homotopy: $\pi_*(\mathrm{colim}\, X_i) = \mathrm{colim}\, \pi_*(X_i).$

$X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots$

$$\mathrm{colim}\begin{pmatrix} X_0 \hookrightarrow X_1 \\ \quad\quad\quad \wr \\ \quad\quad\quad X_2 \end{pmatrix} = X_2$$



$\pi_n(\mathrm{colim}\, X_i)$
$= [S^n, \mathrm{colim}\, X_i]$

Lemma $S^n \to \operatorname{colim} X_i$ factors as

$$\begin{tikzcd} S^n \arrow[rr] \arrow[dr, dotted] & & \operatorname{colim} X_i \\ & X_m \arrow[ur] & \end{tikzcd}$$

for a large m. (Also $S^n \times I$ is compact)

S^n is compact

$\Rightarrow \left(\operatorname{Im}(S^n) \cap (X_m \setminus X_{m-1}) \right) \neq 0$. happens for only finitely many m

---

Pf: ① $\operatorname{colim} H_*(X_i)$

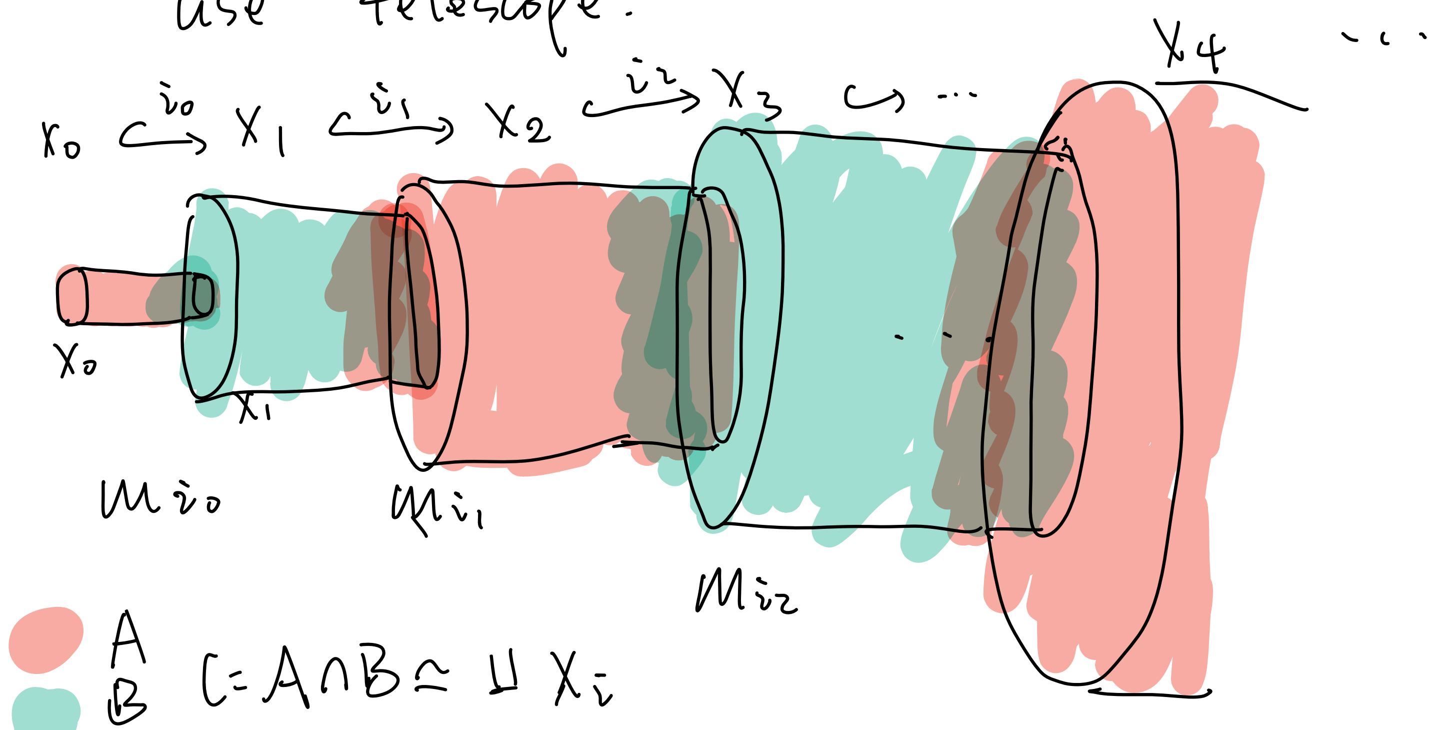
$= \operatorname{colim} \left(\bigoplus H_*(X_i) \underset{\text{inc}}{\overset{\text{id}}{\rightrightarrows}} \bigoplus H_*(X_i) \right)$.

Another description of sequencial colim in Ab.

② $H_*(\operatorname{colim} X_i)$.

use telescope.

$X_0 \overset{i_0}{\hookrightarrow} X_1 \overset{i_1}{\hookrightarrow} X_2 \overset{i_2}{\hookrightarrow} X_3 \hookrightarrow \cdots$



$C = A \cap B \simeq \coprod X_i$

$$A \simeq \coprod_{i \text{ even}} X_i \qquad X = \operatorname{colim} X_i.$$

$$B \simeq \coprod_{i \text{ odd}} X_i$$

$$\to H_*(C) \to \begin{array}{c} H_*(A) \\ \oplus \\ H_*(B) \end{array} \to H_*(X) \to \cdots$$

$$\to \bigoplus_{\text{all}} H_*(X_i) \xrightarrow{(i^A{}_*,\ i^B{}_*)} \begin{array}{c} \bigoplus_{\text{even}} H_*(X_i) \\ \oplus \\ \bigoplus_{\text{odd}} H_*(X_i) \\ \| \\ \bigoplus_{\text{all}} H_*(X_i) \end{array} \xrightarrow{\begin{pmatrix} i_{A*} \\ -i_{B*} \end{pmatrix}} H_*(X) \to$$

this mv sequence compute $H_*(X)$

$$\begin{array}{c} H_*(X) \\ \| \\ \operatorname{colim} H_*(X_i) \underset{\text{(lemma)}}{=} \operatorname{colim}\left(\begin{array}{c} \bigoplus H_*(X_i) \\ \downdownarrows \\ \bigoplus H_*(X_i) \end{array} \right) \end{array}$$

- $\tilde{H}$ v.s. H

H: cat of pairs $(X, A) \to Ab^{gr}$

$\tilde{H}$: pted spaces $* \hookrightarrow X \to Ab^{gr}$.

$$H_*(X) = H_*(X, \emptyset)$$
$$\tilde{H}_*(X) = H_*(X, *).$$

difference: $H_*(\emptyset) \hookrightarrow H_*(*)$ $=$ a $\mathbb{Z}$ in deg 0.

- axioms for H

exactness / additivity / w.e. / dim / excision

$$H_*(A) \to H_*(X) \to H_*(X, A) \xrightarrow{\partial} H_{*-1}(A) \to$$

- axioms for $\tilde{H}$

exactness $A \xrightarrow{i} X$ cof

$$\tilde{H}_*(A) \to \tilde{H}_*(X) \to \tilde{H}_*(X/A)$$

$\tilde{H}_{*-1}(A) \cong \tilde{H}_*(\Sigma A)$

$\tilde{H}_*(X/A) = \tilde{H}_*(Ci) \to \tilde{H}_*(\Sigma A)$

$X \to Ci \to \cdot \cong \Sigma A$

suspension

$$\tilde{H}_*(X) \xrightarrow[\cong]{\Sigma} \tilde{H}_{*+1}(\Sigma X)$$

dim $\tilde{H}_*(S^0) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{o.w.} \end{cases}$, $S^0 = * \; *$

add $\tilde{H}_*(\vee X_i) \cong \oplus \tilde{H}_*(X_i)$

$$H_*(X) = \tilde{H}_*(X_+) = H_*(X_+, +) = \tilde{H}_*(X) \oplus \underbrace{H_*(+,+)}_{=0}$$

$X_+ = X \amalg$ base point.

dim + susp : $\tilde{H}_*(S^n) = \begin{cases} \mathbb{Z} & * = n \\ 0 & * \neq n. \end{cases}$

- see uniqueness using $\tilde{H}$
- for each cell : $H_*(\delta^n) = \begin{cases} \mathbb{Z} & * = n \\ 0 & \text{o.w.} \end{cases}$
- glue :

$$\begin{tikzcd} \vee S^n \arrow[r] \arrow[d, hook] & X_n \arrow[d] \\ \vee D^{n+1} \arrow[r] & X_{n+1} \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

$$\begin{tikzcd} \delta^n \arrow[d, hook, "i"'] \\ D^{n+1} \arrow[d] \\ Ci = S^{n+1} \end{tikzcd}$$

- Hurewicz map

$$\begin{tikzcd} \pi_n(S^n) \arrow[r] \arrow[d, equal] & \tilde{H}_n(S^n) \arrow[d, equal] \\ \mathbb{Z} & \mathbb{Z} \end{tikzcd}$$

Def Hurewicz map

$$\pi_n(X) \xrightarrow{h.} \tilde{H}_n(X)$$

$$\begin{array}{ccc} \pi_n(X) & & \tilde{H}_n(X) \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ x : S^n \to X & \longmapsto & y \end{array}$$

$y = x_*(1)$ ← generator of $\tilde{H}_n(S^n)$

$$\begin{tikzcd} x_* : \tilde{H}_n(S^n) \arrow[r] \arrow[d, equal] & \tilde{H}_n(X) \\ \mathbb{Z} \ni 1 \arrow[r, mapsto] & y \end{tikzcd}$$

check:

1). well definedness

$\gamma' \sim \gamma : S^n \to X$

$\Rightarrow \gamma'_* = \gamma_* : \tilde{H}_n(S^n) \to \tilde{H}_n(X)$

w.e axiom

2). h is a grp homomorphism

Th'm (Hurewicz).

X is $(n-1)$-connective, then

$h : \pi_*(X) \to \tilde{H}_*(X)$

is an equivalence at $* = n$.

e.g. S^n $(n-1)$-connective.

$\pi_n(S^n) = \mathbb{Z} \xrightarrow[h]{\cong} \tilde{H}_*(S^n) = \mathbb{Z}$

CW construction — Aside

unreduced: $X_0 = \coprod pt.$

reduced.

$X_{-1} = *$

X_0 : attaching 0-cells to X_{-1} $= \vee S^0$

Whitehead th'm If $f: X \to Y$ map of CW-cplx and $\pi_*(f)$ is iso $\Rightarrow$ X is h.e. to Y

Whitehead th'm + **Hurewicz th'm**

$\Uparrow$

π_* approximates homotopy type

$\Uparrow$

$\tilde{H}_*$ approximates π_*

$\rightsquigarrow$ **Th'm** (Whitehead th'm)

X, Y CW cplx simply connective. / 1-connective

$f: X \to Y$ s.t. $\tilde{H}_*(f)$ iso

$\Rightarrow$ X h.e. to Y.

Nov 6.

$$h: \pi_n(X) \longrightarrow \tilde{H}_n(X)$$

1) compatible with Σ, i.e.

$$\begin{tikzcd} \pi_n(X) \arrow[r, "h"] \arrow[d, "\Sigma"'] \arrow[dr, phantom, "\circlearrowright"] & \tilde{H}_n(X) \arrow[d, "\Sigma", "\cong"'] \\ \pi_{n+1}(\Sigma X) \arrow[r, "h"] & \tilde{H}_{n+1}(\Sigma X) \end{tikzcd}$$

$\underset{\text{unit}}{1_n} \in \tilde{H}_n(S^n) \cong \mathbb{Z}$

$\underset{\text{unit}}{1_{n+1}} \in \tilde{H}_{n+1}(S^{n+1}) \cong \mathbb{Z}$

$$\begin{array}{ccc} f \in \pi_n(X) & \longmapsto & h(f) = f_*(1_n) \\ \Sigma \downarrow & & \\ \Sigma f & \longmapsto & h(\Sigma f) = (\Sigma f)_*(1_{n+1}) \end{array}$$

$$\overset{\Sigma}{\longrightarrow}$$

$$\parallel$$

$$(\Sigma f)_*(\Sigma 1_n) = \Sigma(f_*(1_n))$$

2) homomorphism

wts: $a, b \in \pi_n(X)$

$h(a+b) = h(a) + h(b)$

$a, b \in \pi_n X$ what is $a+b \in \pi_n X$?

$S^n \xrightarrow{\text{pinch}} S^n \vee S^n \xrightarrow{a \vee b} X$

$(a+b)_*$: apply $\tilde{H}_n$ to

$\tilde{H}_n(S^n) \to \tilde{H}_n(S^n \vee S^n) \to \tilde{H}_n(X)$

$\mathbb{Z} \quad (1,1) \quad \mathbb{Z} \oplus \mathbb{Z} \quad \begin{pmatrix} a_* \\ b_* \end{pmatrix}$

$h(a+b) = (a+b)_*(1_n)$

$= a_*(1_n) + b_*(1_n)$

$= h(a) + h(b)$

Thm ① $h : \pi_1(X) \to \tilde{H}_1(X)$ is an abelianization.

② X $(n-1)$-connective $n \geq 2$

then $h : \pi_n(X) \to \tilde{H}_n(X)$ is an equivalence.

Aside

X 2-connective

① $H_1 \simeq Ab(\pi_1) = 0$

② $H_2 \simeq \pi_2 = 0$

③ $H_3 \simeq \pi_3$

① $\Rightarrow Ab(\text{free grp}(S)) \cong \ast_S \mathbb{Z} \leftarrow \pi_1(\vee_S S^1)$

$\quad = \text{free Ab grp}(S)$

$\quad = \oplus_S \mathbb{Z} \leftarrow \tilde{H}_1(\vee_S S^1)$

pf. π_* : n-cells don't change $\pi_{\leq n-2}$

$\tilde{H}_*$: n-cells don't change $\tilde{H}_{\leq n-2}$ & $\tilde{H}_{\geq n+1}$

$\Downarrow$

exact sequence.

$$\begin{array}{ccc} S^{n-1} & \to & X \\ \downarrow & & \downarrow \\ D^n & \to & X' \end{array}$$

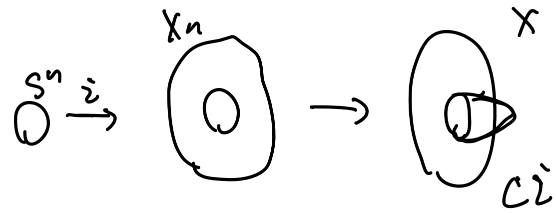
$\to \tilde{H}_*(X) \to \tilde{H}_*(X') \to \tilde{H}_*(X'/X) \to$

$\qquad\qquad\qquad\qquad \tilde{H}_*(X'/X) \cong \tilde{H}_*(S^n)$

$\tilde{H}_*(S^n) = \begin{cases} \mathbb{Z} & * = n \\ 0 & \end{cases}$ $\leadsto$ affect only n & $n-1$

to show ② $\Rightarrow$ suffice to assume X consists of n & $n+1$ cells.

$$
\begin{tikzcd}
L = \vee S^n \arrow[r, "f"] \arrow[d, hook] & \vee S^n = X_n \arrow[d] \\
\vee D^{n+1} \arrow[r] & X = X_{n+1}
\end{tikzcd}
$$



$$
\begin{tikzcd}[column sep=small]
\text{line 1} & \arrow[r] & \pi_n(L) \arrow[r] \arrow[d, "h"] & \pi_n(X_n) \arrow[r] \arrow[d, "h"] & \pi_n(X_{n+1}) \arrow[r] \arrow[d, "h"] & \pi_{n-1}(\vee S^n) = 0 \arrow[d, "h"] \\
\text{line 2} & \arrow[r] & \tilde{H}_n(L) \arrow[r] & \tilde{H}_n(X_n) \arrow[r] & \tilde{H}_n(X_{n+1}) \arrow[r] & 0
\end{tikzcd}
$$

$\tilde{H}_n(X_{n+1}) = \tilde{H}_n(Cf)$, $\quad 0 = \tilde{H}_{n-1}(L) = \tilde{H}_{n-1}(\vee S^n)$

apply π_n / $\tilde{H}_n$ to $L \to X_n \to X_{n+1}$

line 2 is exact : $X = X_{n+1}$ is Cf + exactness axiom

line 1 is exact : L, X_n : $(n-1)$ connective.

+ homotopy excision

both h are eq. $\Rightarrow$ h is an eq.

Aside

π_n : long exact sequence associated to fiber seq

$F \to E \to B$

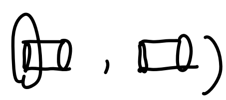
$\cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \cdots$

In general, no such thing for cofiber seq.

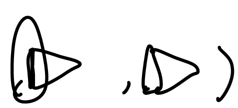
H_n : ✓ for cofiber seq.

$A \xrightarrow{i} X \to Ci$

$\to \pi_n(A) \to \pi_n(X) \to \pi_n(X, A) \to \pi_{n-1}(A) \to$

$\pi_n(X,A) = \pi_n($  $)$

$=?$ $\Longleftrightarrow$ excision

$\pi_n($  $)$

Nov 8

dimension axiom

$$\tilde{H}_*(S^0) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{otherwise} \end{cases} \rightsquigarrow \text{replace } \mathbb{Z} \text{ with } \pi$$

Ab grp. π coefficient

cellular homology with coefficient π

$$C_n(X) = H_n(X^n, X^{n-1}) \cong \tilde{H}_n(\underbrace{X^n/X^{n-1}}_{\text{wedge sum of many copies of } S^n})$$

direct sum of many copies of $\mathbb{Z}$

$\pi = \mathbb{Z}$:

$$C_n(X) \xrightarrow{d_n} C_{n-1}(X) :$$

$C_n(X) = H_n(X^n, X^{n-1})$, $C_{n-1}(X) = H_{n-1}(X^{n-1}, X^{n-2})$

$$H_n(X^n, X^{n-1}) \xrightarrow{\partial} H_{n-1}(X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$$

$(X^{n-1}, \phi) \hookrightarrow (X^{n-1}, X^{n-2})$

equivalently

$$C_n(X) \xrightarrow{d_n} C_{n-1}(X)$$

$C_n(X) = \tilde{H}_n(X^n/X^{n-1})$, $C_{n-1}(X) = \tilde{H}_{n-1}(X^{n-1}/X^{n-2})$

$$\tilde{H}_n(X^n/X^{n-1}) \to \tilde{H}_n(\Sigma X^{n-1}) \cong \tilde{H}_{n-1}(X^{n-1}) \to \tilde{H}_{n-1}(X^{n-1}/X^{n-2})$$

X^n/X^{n-1}

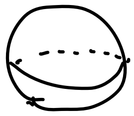
$= \;\; \xrightarrow[\text{out}]{\text{quoient}} \Sigma X^{n-1}$

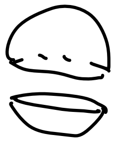
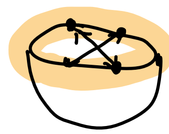
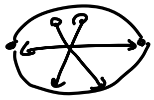
For other π :

$$H_*(C_*(X) \otimes \pi) = H_*^{\text{cell}}(X ; \pi)$$

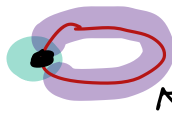
homology of the chain cplx. / homological alg

cellular homolgy of a space. / algebraic topology

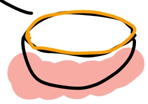
$X = \mathbb{RP}^2$  / antipodal pts.

 glue $\rightsquigarrow$  $\longrightarrow$ boundary = 

$= S^1$

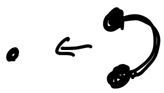
CW decomposition of $\mathbb{RP}^2$: S^1 

deg 2 map $S^1 \to S^1$



glue one 2-cell

$$\begin{array}{ccccc} \text{0-cell} & & \text{1-cell} & & \text{2-cell} \\ \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \xleftarrow{\times 2} & \mathbb{Z} \end{array}$$



$$H_*^{cell}(\mathbb{RP}^2;\mathbb{Z}) = \begin{array}{ccc} 0 & 1 & 2 \\ \mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 0 \end{array}$$

$$H_*^{cell}(\mathbb{RP}^2;\mathbb{Z}/2) = H_*\left(\mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2 \xleftarrow{\times 2 \,=\, 0} \mathbb{Z}/2\right)$$

$$= \begin{array}{ccc} 0 & 1 & 2 \\ \mathbb{Z}/2 & \mathbb{Z}/2 & \mathbb{Z}/2 \end{array}$$

$$H_*^{cell}(\mathbb{RP}^2;\mathbb{Z}/3) = H_*\left(\mathbb{Z}/3 \xleftarrow{0} \mathbb{Z}/3 \xleftarrow{\times 2} \mathbb{Z}/3\right)$$

$$= \begin{array}{ccc} 0 & 1 & 2 \\ \mathbb{Z}/3 & 0 & 0 \end{array}$$

Uniqueness th'm

$H_*(X, A)$ axiomatic homology theory (any homology theory satisfying axioms)

$H_*^{cell}(X, A)$ cellular homology

Th'm $\exists$ map: $H_*(X, A) \cong H_*^{cell}(X, A)$ & $\cong$ compatible with ∂.

$0 \to C_*(A) \to C_*(X) \to C_*(X, A) \to 0 \rightsquigarrow$ LES & ∂

Pf

using reduced homology.

reduced homology $\overset{\text{determine}}{\longleftrightarrow}$ unreduced homology

$\rightsquigarrow$ $H(X, A) := \tilde{H}(X/A)$.

$\tilde{H}(X) := H(X, *)$ $\leftsquigarrow$

uniqueness (Th'm for reduced homology)

$\exists$ map : $\tilde{H}_*(X) \cong \tilde{H}_*^{cell}(X)$

& compatible with Σ

$\tilde{H}_*^{cell}(X) := H_*(\tilde{C}_\cdot(X))$

$\tilde{C}_n(X) = \tilde{H}_*(X^n/X^{n-1})$

$\tilde{C}_0(X) = \tilde{H}_*(X^0/X^{-1})$ differs by a $\mathbb{Z}$

$C_0(X) = H_*(X^0/X^{-1})$

Pf. 1) show $\cong$ for each H_n.

use. the fact : H_n only depends on n-cells & $(n+1)$-cells

$\Rightarrow$ reduce to : X consists of n-cells . $(n+1)$-cells

$$\begin{tikzcd} X/X^{n-1} = \vee S^{n-1} \arrow[r] \arrow[d, hook] & X^{n-1} \arrow[d] \\ \vee D^n \arrow[r] & X \end{tikzcd}$$

$\leftrightsquigarrow$ • $X/X^{n-1} \to X^{n-1} \to X$ cofiber sequence

• $X^{n-1} = \vee S^{n-1}$

By axioms: ① $X = S^n$ ✓ (dim + suspension)

② $X = \vee S^n$ ✓ (① + add)

③

$$\begin{tikzcd} H_n(X/X^{n-1}) \arrow[r, "\partial"] \arrow[d, "\cong"] & H_{n-1}(X^{n-1}) \arrow[r] \arrow[d, "\cong"] & H_{n-1}(X) \arrow[r] & H_{n-1}(X/X^{n-1}) \\ H_n^{cell}(X/X^{n-1}) \arrow[r, "\partial"] & H_{n-1}^{cell}(X^{n-1}) \arrow[r] & H_{n-1}^{cell}(X) \arrow[r] & 0 \end{tikzcd}$$

$H_{n-1}(X/X^{n-1}) = H_{n-1}(S^n) = 0$

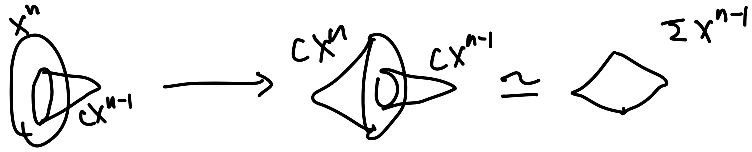
$H_{n-1}(X) = \operatorname{coker} \partial$

have $\cong$ by ① & ②, remains to show ?

$\cong$

to show ↻ : need to unpack def of ∂ for H_* & H^{cell}_*

both ∂ are induced by



naturality $\rightsquigarrow$ ↻

2) Check compatibility with Σ

by comparing $\pi_n(X)$ using Hurewicz map.

(uniqueness with coeff)

$$H^{cell}_*(X, A; \pi) \cong H_*(X, A)$$

satisfying dim axiom

$$H_*(pt) = \begin{cases} \pi & * = 0 \\ 0 & \text{o.w.} \end{cases}$$

---

some homological alg.

$C_*(X)$

$H_*(C_*(X)) \otimes \mathbb{Z}/2$ — can be different. e.g. $\mathbb{RP}^2$ case.

$H_*(C_*(X) \otimes \mathbb{Z}/2)$

measure the difference $\rightsquigarrow$ can compute $\mathbb{Z}/2$ coeff from $\mathbb{Z}$ coeff.

Reason for being different:

$$0 \to A \to B \xrightarrow{f} C \to 0.$$

$$A \otimes M \to B \otimes M \to C \otimes M \to 0.$$

$-\otimes M$ is a right exact funcor

$A = \ker f.$

$A \otimes M \neq \ker(f \otimes M)$

failure. measured by right derived functor of $\otimes M$

$\mathrm{Tor}^{\mathbb{Z}}_{\bullet}(-, M)$

$\mathrm{Tor}_2^{\mathbb{Z}}(C,M) \leftarrow \cdots$

$$\Rightarrow \mathrm{Tor}_1^{\mathbb{Z}}(A,M) \to \mathrm{Tor}_1^{\mathbb{Z}}(B,M) \to \mathrm{Tor}_1^{\mathbb{Z}}(C,M) \to A \otimes M \to B \otimes M \to C \otimes M \to 0$$

$\mathrm{Tor}^{\mathbb{Z}}_{\bullet}(X, M)$ definition:

1) find a ~~flat~~ free resolution of X : $C_\bullet$

Eg: $X = \mathbb{Z}/2$ $\quad H_*\left(\begin{array}{c}\mathbb{Z}\\ \downarrow \times 2\\ \mathbb{Z}\end{array}\right) = \begin{array}{c}0\\ \cdot\\ \mathbb{Z}/2\end{array} = X$

<u>resolusion</u>: a chain cplx $C_\bullet$ with $H_*(C_\bullet) = X$

viewed as <u>concentrated in deg 0</u>

<u>over $\mathbb{Z}$</u>: always resolve in 2 step.

$$H_*(\mathbb{Z}/p) = H_*\left(\begin{array}{cc}0 & 3\\ \downarrow & \\ 0 & 2\\ \downarrow & \\ \mathbb{Z} & 1\\ \downarrow \times p & \\ \mathbb{Z} & 0\end{array}\right)$$

X : abelian grp.

viewed as chain cplx

$$\begin{array}{cc}2 & 0\\ & \downarrow\\ 1 & 0\\ & \downarrow\\ 0 & X\end{array}$$

2) $C_\bullet \otimes M$

3) $\mathrm{Tor}_n^{\mathbb{Z}}(C, M) = H_n(C_\bullet \otimes M)$