1. Review of Electrostatics

Coulomb's law

$$\underset{\vec{r}_1}{q_1 \bullet} \quad \xleftarrow{\;\vec{r}_{12}\;} \quad \underset{\vec{r}_2}{\bullet\, q_2}$$

$$\vec{r}_{12} = \vec{r}_1 - \vec{r}_2$$

$$r_{12} = |\vec{r}_1 - r_2|$$

$$\hat{r}_{12} = \frac{\vec{r}_{12}}{r_{12}}$$

Force on charge 1 at rest from charge 2 at rest

$$\vec{F}_{12} = k_1 \frac{q_1 q_2}{r_{12}^3} \vec{r}_{12}$$

Force on charge Q at $\vec{R}$

$$\vec{F}_Q = Q k_1 \sum_i \frac{q_i (\vec{R} - \vec{r}_i)}{|\vec{R} - \vec{r}_i|^3}$$

Introduce charge density

$$\rho(\vec{r}) = \sum_i q_i \, \delta^3(\vec{r} - \vec{r}_i)$$

Then

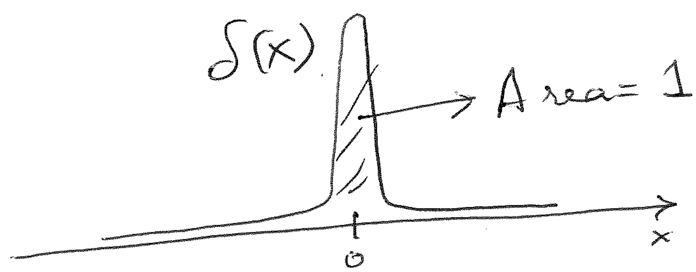
$$\vec{F}_Q = k_1 Q \int d^3r \, \frac{\rho(\vec{r}) \, (\vec{R} - \vec{r})}{|\vec{R} - \vec{r}|^3}$$

Review of δ functions

$$\int dx\, f(x)\,\delta(x - x_0) = f(x_0)$$

$$\delta^3(\vec{r}) = \delta(x)\,\delta(y)\,\delta(z)$$

$$\delta(x) = \lim_{a \to 0} \frac{1}{\sqrt{\pi a}}\, e^{-x^2/a}$$



$$= \lim_{\epsilon \to 0} \frac{1}{\pi} \frac{\epsilon}{\epsilon^2 + x^2}$$

Fourier Thm.

If $f(x) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} dk\, f_k\, e^{ikx}$

then $f_k = \displaystyle\int_{-\infty}^{\infty} dx\, f(x)\, e^{-ikx}$

So $f(x) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} dk\, e^{ikx} \int_{-\infty}^{\infty} dx'\, f(x')\, e^{-ikx'}$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx'\, f(x') \left[\int_{-\infty}^{\infty} dk\, e^{ik(x-x')} \right]$$

$$\Rightarrow \boxed{\delta(x - x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ik(x-x')}}$$

Introduce <u>electric field</u>

$$\vec{E}(\vec{R}) = \lim_{Q\to 0} \frac{F_Q(\vec{R})}{Q}$$

$$\boxed{E(\vec{R}) = k_1 \int d^3r \, \frac{\rho(\vec{r})\,(\vec{R}-\vec{r})}{|\vec{R}-\vec{r}|^3}}$$

Equivalent $\boxed{\vec{\nabla}\cdot\vec{E} = 4\pi k_1 \rho(\vec{r})}$

Use
$$\nabla_{\vec{R}}\left(\frac{1}{|\vec{R}-\vec{r}|}\right) = \hat{x}\frac{\partial}{\partial X}\left(\frac{1}{\sqrt{(X-x)^2+(Y-y)^2+(Z-z)^2}}\right) + \cdots$$

$$= -\frac{(X-x)\hat{x} + \cdots}{|\vec{R}-\vec{r}|^3}$$

$$= -\frac{\vec{R}-\vec{r}}{|\vec{R}-\vec{r}|^3}$$

So $\vec{E}(\vec{R}) = -\vec{\nabla}_R \Phi(\vec{R})$

$$\Phi(\vec{R}) = \int d^3r \, \frac{\rho(\vec{r})}{|\vec{R}-\vec{r}|}$$ is the electrostatic potential

So

$$\nabla \cdot E(r) = -k_1 \int d^3r' \, \rho(\vec{r}\,') \, \nabla_r^2 \frac{1}{|\vec{r}-\vec{r}\,'|}$$

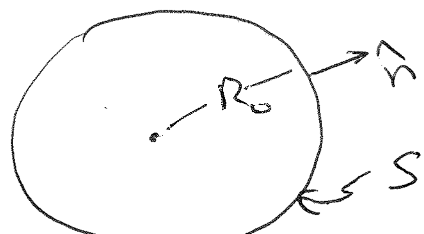
Key result

$$\boxed{\nabla_r^2 \frac{1}{|\vec{r}|} = -4\pi \delta(\vec{r})}$$

For $\vec{r} \neq 0$ verify by explicit computation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \frac{1}{\sqrt{x^2+y^2+z^2}} = 0.$$

For $\vec{r} = 0$, integrate over a sphere of radius R_0 around the origin



$$\int_S d^3r \, \nabla^2\left(\frac{1}{r}\right) = \int_S d^3r \, \vec{\nabla} \cdot \vec{\nabla}\left(\frac{1}{r}\right)$$

$$= \int_S d^3r \, \vec{\nabla} \cdot \left(-\frac{\vec{r}}{r^3}\right)$$

Using Gauss' Theorem

$$= -\int_S da \, \frac{\hat{n} \cdot \vec{r}}{r^3} = -\frac{4\pi R_0^2}{R_0^2}$$

$$= -4\pi.$$

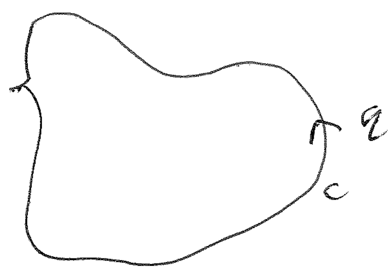
So then $\vec{\nabla} \cdot \vec{E} = 4\pi k_1 \rho(\vec{r})$

Differential form of Coulomb's law.

Also valid with time dependence

Also because
$$\vec{E} = -\vec{\nabla}\Phi$$

$$\vec{\nabla} \times \vec{E} = 0.$$



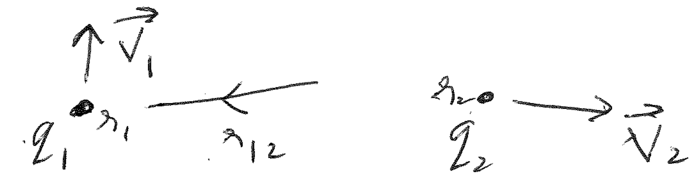
Electric forces are conservative.

No work is done in transporting a charge around a closed loop

$$\oint_C \vec{E} \cdot d\ell = 0$$

by Stokes Thm.

2. Magnetostatics



Biot - Savart Law

$$\vec{F}_{12} = k_2\, q_1 \vec{V}_1 \times \left\{ q_2 \frac{\vec{V}_2 \times \vec{r}_{12}}{|r_{12}|^3} \right\}.$$

Depends upon the velocities (~~parall~~ parallel currents attract)

By analogy with electrostatics, we introduce a magnetic field

$$\vec{B}(\vec{r}) = k_3 \sum_i q_i \frac{\vec{V}_i \times (\vec{r} - \vec{r}_i)}{|\vec{r} - \vec{r}_i|^3}$$

Then force on a test charge Q at $\vec{R}$ with velocity $\vec{V}$

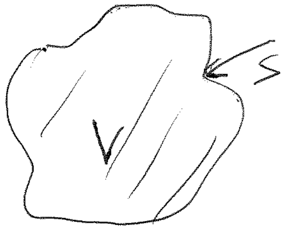
$$\Rightarrow \boxed{F_Q(\vec{R}) = \frac{k_2}{k_3} Q\, \vec{V} \times \vec{B}(\vec{R})}$$

Now we introduce a current density

$$\vec{J}(\vec{r}) = \sum_i q_i \vec{V}_i\, \delta(\vec{r} - \vec{r}_i).$$

Then

$$\vec{B}(\vec{r}) = k_3 \int d^3r' \, \frac{\vec{J}(\vec{r}\,') \times (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3}$$



Constraint on current density:

Rate of change of charge density inside volume $V = -$ (currents leaving volume)

$$\frac{d}{dt}\int_V d^3r \, \rho(r) = -\int_S \vec{J} \cdot \vec{n} \, da$$

$$= -\int_V \vec{\nabla} \cdot \vec{J} \, d^3r .$$

$$\Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0}$$

Verify $\quad \rho = \sum_i q_i \, \delta^3(\vec{r} - \vec{r}_i)$

$$\frac{\partial \rho}{\partial t} = -\sum_i q_i \frac{d\vec{r}_i}{dt} \cdot \vec{\nabla} \delta(\vec{r} - \vec{r}_i(t))$$

$$\vec{J} = \sum_i q_i \vec{v}_i \, \delta^3(\vec{r} - \vec{r}_i)$$

$$\vec{\nabla} \cdot \vec{J} = \sum_i q_i \vec{v}_i \cdot \vec{\nabla} \delta^3(\vec{r} - \vec{r}_i)$$

Hence $\quad \frac{\partial \rho}{\partial t} = -\nabla \cdot \vec{J}$

For magnetostatics $\underline{\vec{\nabla}\cdot\vec{J}=0}$

$$\vec{B}(\vec{r}) = k_3 \int d^3r' \, \frac{\vec{J}(r') \times (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

Claim $$= k_3 \, \vec{\nabla} \times \int d^3r' \, \frac{\vec{J}(r')}{|\vec{r}-r'|}.$$

Pf. Use $\vec{\nabla}\times(f\vec{g}) = f(\vec{\nabla}\times\vec{g}) + \vec{\nabla f}\times\vec{g}$

Then

$$\vec{\nabla} \times \int d^3r' \, \frac{\vec{J}(\vec{r}')}{|\vec{r}-r'|}$$

$$= \int d^3r' \, \vec{\nabla}\left(\frac{1}{|\vec{r}-r'|}\right) \times \vec{J}(r')$$

$$= \int d^3r' \, \frac{J(\vec{r}') \times (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$ Q.E.D.

~~Using~~ So we can ~~d~~ write

$$\vec{B}(\vec{r}) = k_3 \, \vec{\nabla} \times \int d^3r' \, \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$\Rightarrow \boxed{\nabla \cdot B = 0}$$

Use

$$\nabla \times (\nabla \times \vec{a}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{a}) - \nabla^2 \vec{a}$$

Then evaluate $\vec{\nabla} \times \vec{B}$

First term yields

$$k_3 \int d^3x' \; \vec{\nabla}\left[\vec{\nabla}\cdot\left(\frac{J(x')}{|\vec{x}-x'|}\right)\right]$$

$$= k_3 \vec{\nabla} \int d^3x' \; \vec{J}(\vec{x}') \cdot \vec{\nabla}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)$$

$$= k_3 \vec{\nabla} \int d^3x' \; \vec{J}(\vec{x}') \cdot -\vec{\nabla}'\left(\frac{1}{|\vec{x}-x'|}\right)$$

Integrate by parts

$$= k_3 \vec{\nabla} \int d^3x' \left(\nabla' \cdot J(x')\right) \frac{1}{|\vec{x}-x'|}$$

$= 0$ in magnetostatics.

Second term yields

$$\boxed{\vec{\nabla} \times \vec{B} = -k_3 \int d^3x' \; \nabla^2\left(\frac{J(x')}{|\vec{x}-\vec{x}'|}\right) = 4\pi k_3 \vec{J}(\vec{x})}$$

(using $\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(r)$)

Ampere's Law.