Waves in Matter

$$\nabla \cdot D = 0 \qquad \nabla \times H = \frac{\partial D}{\partial t}$$

$$\nabla \cdot B = 0 \qquad \nabla \times E = -\frac{\partial B}{\partial t}$$

$$D = \epsilon E \; ; \quad B = \mu H.$$

Just have to replace $\epsilon_0 \to \epsilon$, $\mu_0 \to \mu$

for velocity of light

$$v = \frac{1}{\sqrt{\mu \epsilon}} < c \quad \text{(usually)}$$

$$n = \frac{c}{v} \geq 1$$ is the index of refraction.

Plane waves

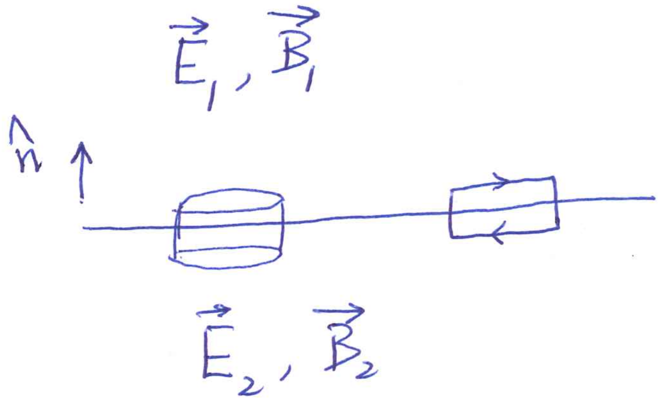
$$\vec{E} = \vec{E}_0 \, e^{i(\vec{k} \cdot \vec{x} - \omega t)} \; ; \quad \vec{B} = \vec{B}_0 \, e^{i(\vec{k} \cdot \vec{x} - \omega t)}$$

$$\omega^2 = v^2 k^2$$

Then $\vec{k} \cdot \vec{E}_0 = 0$, $\vec{k} \cdot \vec{B}_0 = 0$

and $$\vec{B}_0 = \frac{\hat{k} \times \vec{E}_0}{v}$$

Boundary conditions



From Gauss' law on the pillbox

$$\cancel{\hat{n}\cdot(\vec{E}_1 - \vec{E}_2)} \quad \hat{n}\cdot(\vec{D}_1 - \vec{D}_2) = \sigma_{free}$$

surface charge density

$$\hat{n}\cdot(\vec{B}_1 - \vec{B}_2) = 0$$

From Stokes theorem on the loop

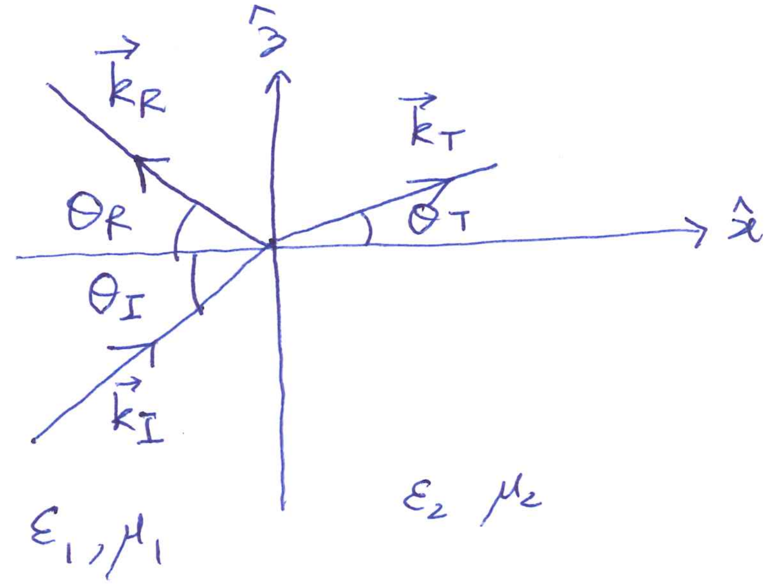
$$\hat{n}\times(\vec{E}_1 - \vec{E}_2) = 0$$

and

$$\hat{n}\times(\vec{H}_1 - \vec{H}_2) = \vec{K}$$

(free) surface current per unit length

Reflection and Refraction



$$\vec{E}_{\text{incident}} = \vec{E}_I\, e^{i(\vec{k}_I\cdot\vec{x}_I - \omega_I t)}$$

$$\vec{k}_I = k_I \cos\theta_I\, \hat{x} + k_I \sin\theta_I\, \hat{z}$$

Similarly for reflected and transmitted waves

$$\vec{k}_R = -k_R \cos\theta_R\, \hat{x} + k_R \sin\theta_R\, \hat{z}$$

$$\vec{k}_T = \vec{k}_T \cos\theta_T\, \hat{x} + k_T \sin\theta_T\, \hat{z}.$$

Now we match at the boundary.

The oscillating phase factors should be equal at the boundary

$$\boxed{\omega_I = \omega_R = \omega_T = \omega}$$

and $\vec{k}_I\cdot\vec{x} = \vec{k}_R\cdot\vec{x} = \vec{k}_T\cdot\vec{x}$ at $x = 0$.

$\Rightarrow$ All $\vec{k}$ lie in x-z plane.

This implies

$$k_I \sin\theta_I = k_R \sin\theta_R = k_T \sin\theta_T$$

~~Also~~ Also $\quad \omega = v_1 k_I = v_1 k_R = v_2 k_T$

This implies $\boxed{\theta_I = \theta_R}$

and

$$\frac{\sin\theta_I}{v_1} = \frac{\sin\theta_2}{v_2}$$

or

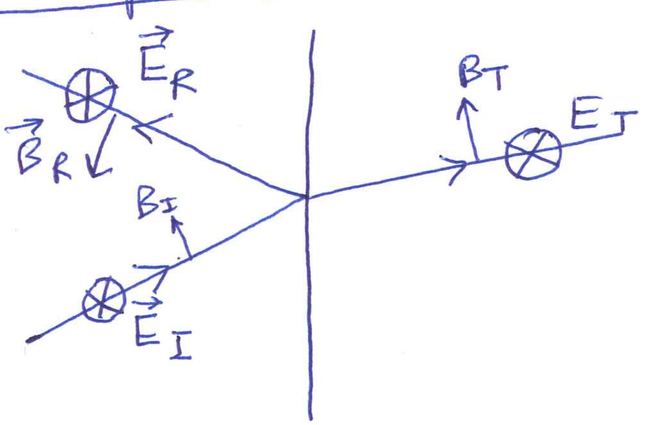
$$\boxed{n_1 \sin\theta_I = n_2 \sin\theta_T}$$

Snell's Law.

Fresnel Equations

The fate of the fields depends upon the polarization

Normal polarization (s-polarization)



$$\vec{E}_I = E_I \hat{y}$$

$$\vec{E}_T = E_T \hat{y}$$

$$\vec{E}_R = E_R \hat{y}$$

Matching tangential $\vec{E}$ fields

$$E_I + E_R = E_T$$

Using $\vec{B} = (\hat{k} \times \vec{E})/v$ and matching normal component of B

$$B_I \cos\theta_I - B_R \cos\theta_R = B_T \cos\theta_T$$

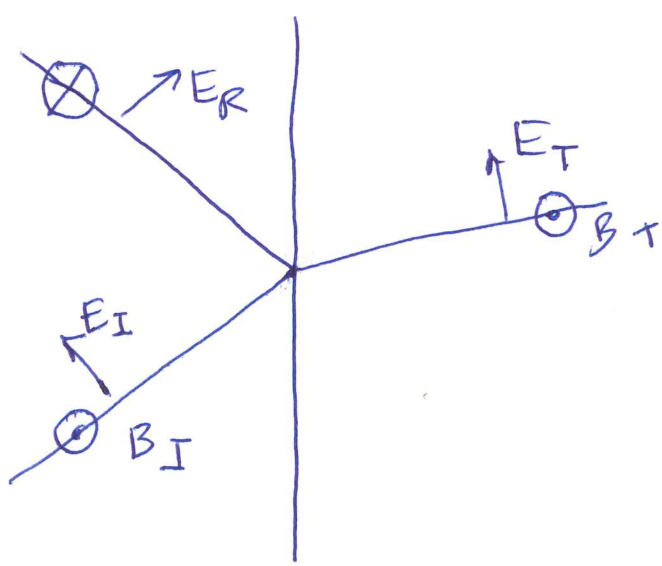
$$\Rightarrow \quad \frac{E_I - E_R}{v_1} \cdot \cos\theta_I = \frac{E_T}{v_2} \cos\theta_T$$

Solving

$$\frac{E_R}{E_I} = \frac{n_1 \cos\theta_I - n_2 \cos\theta_T}{n_1 \cos\theta_I + n_2 \cos\theta_T}$$

$$\frac{E_T}{E_I} = \frac{2 n_1 \cos\theta_I}{n_1 \cos\theta_I + n_2 \cos\theta_T}$$

Parallel Polarization



We have

$$\vec{E}_I \cdot \vec{k} = 0$$

which implies

$$\vec{E}_I = -E_I \sin\theta_I \,\hat{x} + E_I \cos\theta_I \,\hat{z}$$

Matching tangential components of the electric field

$$E_I \cos\theta_I + E_R \cos\theta_R = E_T \cos\theta_T$$

Matching tangential components of the magnetic field

$$B_I - B_R = B_T$$

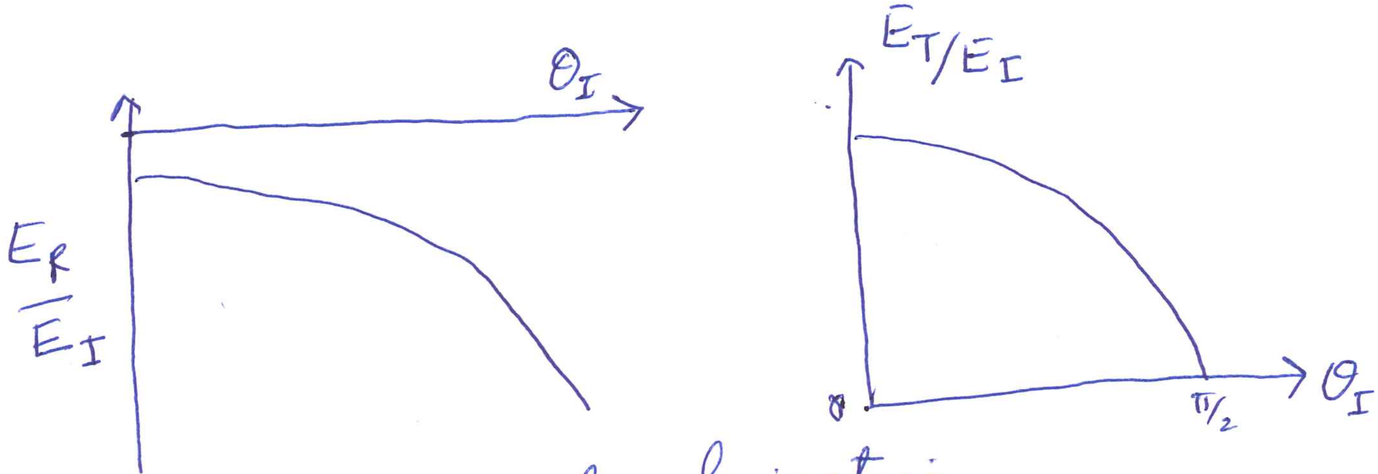
$$\Rightarrow \quad \frac{E_I - E_R}{v_1} = \frac{E_T}{v_2}.$$

Then we obtain the Fresnel equations for parallel polarized light.

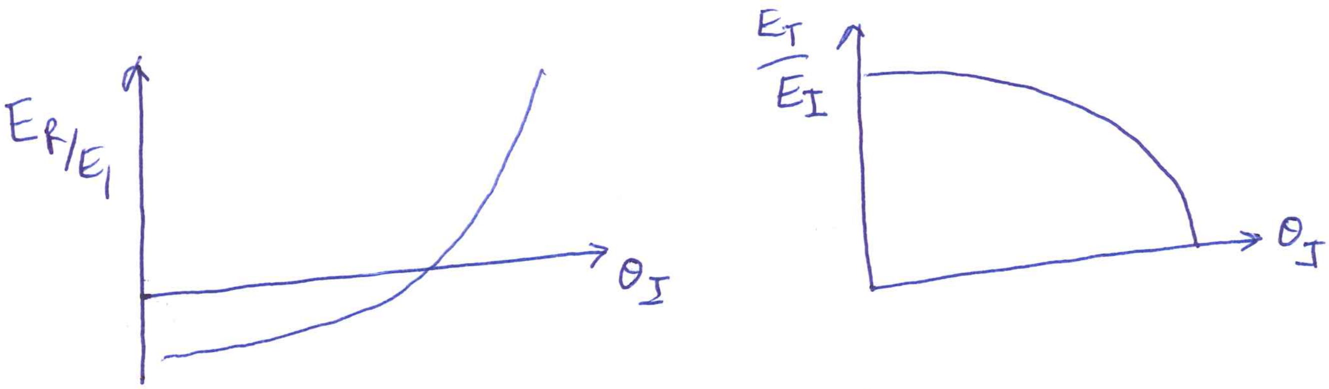
$$\frac{E_R}{E_I} = \frac{n_1 \cos\theta_T - n_2 \cos\theta_I}{n_1 \cos\theta_T + n_2 \cos\theta_I}$$

$$\frac{E_T}{E_I} = \frac{2 n_1 \cos\theta_I}{n_1 \cos\theta_T + n_2 \cos\theta_I}$$

$n_1 = 1 \qquad n_2 = 2$



Normal polarization



Parallel Polarization

Brewster's Angle

$E_R = 0$ for $\tan\theta_B = \dfrac{n_2}{n_1}$

Reflected light at angle θ_B is polarized normal to the plane of incidence and reflection

Total internal reflection

Recall $n_1 \sin\theta_I = n_2 \sin\theta_T$

or $\sin\theta_T = \frac{n_1}{n_2}\sin\theta_I$.

When $\frac{n_1}{n_2}\sin\theta_I > 1$ we have total internal reflection.

Only possible for $n_2 < n_1$
and $\theta > \theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)$.

Recall that

$$k_z = k_I \sin\theta_I = \frac{\omega}{v_1}\sin\theta_I = k_{Tz}$$

Al~~so~~ Maxwell's equation requires

$$\omega^2 = v_2^2 k_T^2$$

So $$k_{Tx} = \sqrt{k_T^2 - k_{Tz}^2}$$

$$= \pm \frac{\omega}{v_2}\sqrt{1 - \frac{v_2^2 \sin^2\theta_I}{v_1^2}}$$

$$= \pm \frac{\omega}{v_2}\sqrt{1 - \underbrace{\frac{n_1^2 \sin^2\theta_I}{n_2^2}}_{>1}}$$

$= $ pure imaginary!

So

$$\vec{E}_{T2} = \vec{E}_T \, e^{i(k_T z - \omega t)} \, \underbrace{e^{-\alpha \omega x / V_2}}_{\text{Evanescent wave in region 2}}$$