Dispersion in Insulators

We have assumed so far that the polarization response is local in both <u>space</u> and <u>time</u>

i.e. $\vec{P}(\vec{r},t) = \epsilon_0 \chi \vec{E}(\vec{r},t)$.

Spatial locality is usually a good approximation because wavelength of light $\gg$ size of molecules.

However locality in time ~~~~ may not be valid because optical frequencies $\simeq$ atomic frequencies.

So we write more generally

$$\vec{P}(\vec{r},t) = \epsilon_0 \int_{-\infty}^{\infty} dt' \, \chi(t-t') \, \vec{E}(\vec{r},t').$$

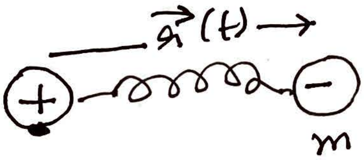
Causality: $\boxed{\chi(t<0) = 0}$

In frequency space

$$P(\vec{r}, \omega) = \epsilon_0 \chi(\omega) \vec{E}(\vec{r}, \omega)$$

Full determination of $\chi(\omega)$ requires quantum mechanics.

We use a simple model → damped harmonic oscillator



$$m \frac{d^2\vec{r}}{dt^2} = -\frac{e\vec{E}}{m} - m\omega_0^2 \vec{r} - m\gamma \frac{d\vec{r}}{dt}$$

↓ frictional force from interacting with other electrons.

In frequency space

$$-m\omega^2 \vec{r}(\omega) = -\frac{e}{m}\vec{E}(\omega) - m\omega_0^2 \vec{r} + i m\gamma \omega \vec{r}(\omega)$$

Solving for $\vec{r}(\omega)$

$$\vec{r}(\omega) = -\frac{e\vec{E}(\omega)}{m} \cdot \frac{1}{(-\omega^2+\omega_0^2 - i\gamma\omega)}$$

So dipole moment

$$\vec{p}(\omega) = \frac{e^2}{m} \frac{1}{(-\omega^2+\omega_0^2-i\gamma\omega)} \vec{E}(\omega).$$

Polarization

$$\vec{P}(\omega) = n\vec{p}(\omega) \qquad n \to \text{density of atoms}.$$

So

$$\chi(\omega) = \frac{ne^2}{\epsilon_0 m} \frac{1}{-\omega^2+\omega_0^2-i\gamma\omega}$$

$$\equiv \frac{\omega_p^2}{-\omega^2+\omega_0^2-i\gamma\omega}.$$

$\omega_p^2 = \frac{ne^2}{\epsilon_0 m}$ is called the "plasma frequency".

Misnomer here, because there are no plasma oscillations in an insulator.

So the ~~pr~~ permittivity

$\epsilon(\omega) = \epsilon_0(1+\chi(\omega))$ is complex and frequency dependent.

Electromagnetic waves

$$\vec{D} = \epsilon(\omega)\vec{E} \quad ; \quad \vec{B} = \mu\vec{H}$$

$$\nabla\cdot D = 0 \Rightarrow \epsilon(\omega)\,\vec{k}\cdot\vec{E} = 0$$

$$\nabla\cdot B = 0 \Rightarrow \vec{k}\cdot\vec{B}(\omega) = 0$$

From the other Maxwell's equations

$$\vec{k}\times\vec{B}(\omega) = -\mu\,\epsilon(\omega)\,\omega\,\vec{E}(\omega)$$

$$\vec{k}\times\vec{E}(\omega) = \omega\,\vec{B}(\omega)$$

(In insulators $|\epsilon(\omega)| \neq 0$ and so we can assume $\vec{k}\cdot\vec{E} = 0$).

So we evaluating $\vec{k}\times(\vec{k}\times\vec{E})$ we obtain as before

$$\vec{k}\cdot\vec{k} = k^2 = \mu\,\epsilon(\omega)\,\omega^2$$

real

complex

complex!

$$k = k_1 + ik_2 = \omega\sqrt{\mu}\,\sqrt{\epsilon_1 + i\epsilon_2}$$

$$V_p = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\,\epsilon(\omega)}}$$; refractive index $n(\omega) = \frac{c}{V_p(\omega)} = \sqrt{\frac{\mu\,\epsilon(\omega)}{\mu_0\,\epsilon_0}}$

"phase velocity"

For a plane wave travelling in the z direction

$$\vec{E}(\vec{r},t) = \vec{E}(\omega)\, e^{-k_2 z} \cos(k_1 z - \omega t)$$

$$\vec{B}(\vec{r},t) = \frac{|k|}{\omega}\left(\hat{z} \times \vec{E}(\omega)\right) \cos(k_1 z - \omega t + \phi) \times e^{-k_2 z}$$

$\vec{E}(\omega) \rightarrow$ vector in x-y plane

$k_2 \rightarrow$ determines <u>attenuation</u> of wave.

$\phi = \tan^{-1}\left(\frac{k_2}{k_1}\right) \rightarrow \vec{B}$ field is not in phase with $\vec{E}$ field.

<u>Wave packets in dispersive media</u> (18.6)

Plane wave z direction, ignore k_2

$$E(z,t) = \int_{-\infty}^{\infty} d\omega\, A(\omega)\, e^{i(k(\omega) z - \omega t)}$$

Assume $A(\omega)$ is peaked near ω_0.

Expand

$$k(\omega) = k(\omega_0) + \left.\frac{dk}{d\omega}\right|_{\omega_0} (\omega - \omega_0) + \frac{1}{2}\frac{d^2k}{d\omega^2}(\omega-\omega_0)^2$$

$$= k_0 + k_0' (\omega - \omega_0) + \frac{1}{2} k_0'' (\omega - \omega_0)^2$$

Then

$$E(z,t) = \underbrace{\int_{-\infty}^{\infty} d\Omega\, A(\omega_0 + \Omega) \exp\left(i\left[\Omega k_0' + \frac{1}{2}\Omega^2 k_0''\right] \times z - i\Omega t\right)}_{A(z,t)} \times \exp(i k_0 z - \omega_0 t).$$

~~Now observe ∂E~~

We are interested in the envelope function.

Note $A(z,t)$ looks like the solution of a Schroedinger eqn of a free particle.

$$\frac{\partial A}{\partial z} = i\Omega k_0' A(z) + \frac{i\Omega^2}{2} k_0'' A(z)$$

$$\frac{\partial A}{\partial t} = -i\Omega A(z)$$

$$\boxed{i\left(\frac{\partial}{\partial z} + k_0' \frac{\partial}{\partial t}\right) A(z,t) = \frac{1}{2} k_0'' \frac{\partial^2 A}{\partial t^2}}$$

First ignore k_0''.

Then solution

$$A(z,t) = f(z - v_g t)$$

where f is any function

and $$v_g = \frac{1}{k_0'} = \frac{d\omega}{dk}\Big|_{k_0}$$

is the group velocity.

In this approximation, pulse travels without distortion.

To include k_0'', change to a time τ co-moving with pulse

$$\tau = t - z/v_g.$$

and write $A(z,t) = \psi(z,\tau)$.

Then
$$i\frac{\partial \psi}{\partial z} = \frac{1}{2} k_0'' \frac{\partial^2 \psi}{\partial \tau^2}$$

(Schroedinger equation with time and space interchanged !!).

For a gaussian wavepacket

$$\psi(0,\tau) = \exp\left(-\tau^2/\Delta T^2\right)$$

$$\psi(z,\tau) = \sqrt{\frac{\Delta T^2}{\Delta T^2 - 2ik_0'' z}}$$

$$\times \exp\left(- \frac{\tau^2}{\Delta T^2 - 2ik_0'' z}\right)$$

$\Rightarrow$

$$|\psi(z,\tau)| = \frac{1}{\left(1 + 2k_0'' z/\Delta T^2\right)^{1/2}}$$

$$\times \exp\left(- \frac{\tau^2}{\Delta T^2 + \frac{(2k_0'' z)^2}{(\Delta T)^2}}\right).$$

"velocity – time" uncertainty

The sharper the initial pulse, the more it broadens with z.

Group and phase velocities

$$\frac{1}{v_g} = \frac{dk_1}{d\omega} = \frac{d}{d\omega}\left(\frac{n\omega}{c}\right) = \frac{1}{v_p} + \frac{\omega}{c}\frac{dn}{d\omega}.$$

$$\frac{dn}{d\omega} > 0 \Rightarrow v_g < v_p \rightarrow \text{normal dispersion}$$

$$\frac{dn}{d\omega} < 0 \Rightarrow v_g > v_p \rightarrow \text{anomalous dispersion}$$

Recall

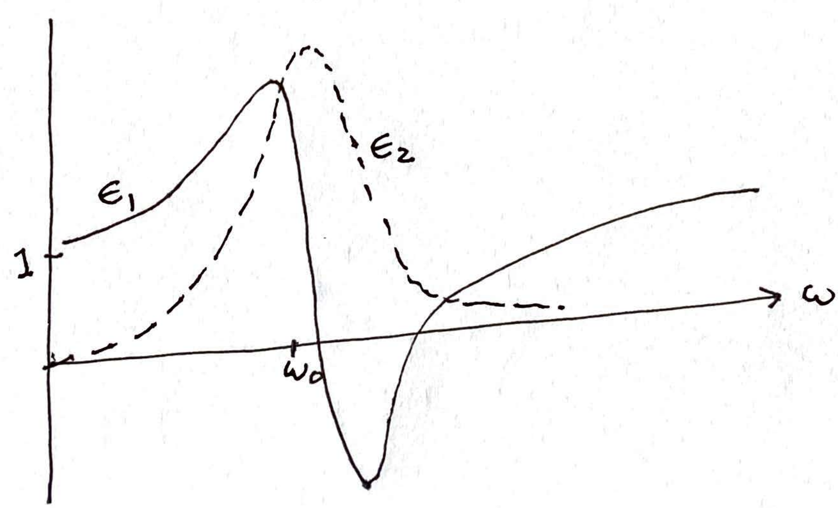
$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{\omega_p^2}{-\omega^2 + \omega_0^2 - i\gamma\omega}$$

So

$$\frac{\epsilon_1(\omega)}{\epsilon_0} = 1 - \frac{\omega_p^2\,(\omega^2 - \omega_0^2)}{(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2}$$

$$\frac{\epsilon_2(\omega)}{\epsilon_0} = \omega_p^2 \frac{\gamma\omega}{(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2}$$

$$\frac{\epsilon_{1,2}(\omega)}{\epsilon_0}$$



~~Also~~ $k^2 = \omega^2 \mu \epsilon(\omega)$ implies

$$k_1 = \pm \omega\sqrt{\mu}\left(\frac{1}{2}\sqrt{\epsilon_1^2+\epsilon_2^2} + \frac{1}{2}\epsilon_1\right)^{1/2}$$

$$k_2 = \pm \omega\sqrt{\mu}\left(\frac{1}{2}\sqrt{\epsilon_1^2+\epsilon_2^2} - \frac{1}{2}\epsilon_1\right)^{1/2}$$

(i) Transparent propagation

$\epsilon_1 > 0$, $\epsilon_1 \gg \epsilon_2$

true for $\omega < \omega_0 - \gamma/2$ and $\omega > \omega^*$

$k_2 << k_1 \rightarrow$ small damping

(ii) Resonant absorption

~~$|\epsilon_2| \gg$~~ $\epsilon_2 \gg |\epsilon_1|$

$$k_1 \approx k_2 \approx \omega\sqrt{\frac{\mu\epsilon_2}{2}}$$

(iv) Total Reflection

$\epsilon_1 < 0$ and $|\epsilon_1| \gg \epsilon_2$

$$k_1 \approx \pm \frac{\epsilon_2}{2}\sqrt{\frac{\mu}{|\epsilon|}}$$

$$k_2 \approx \frac{2|\epsilon_1|}{\epsilon_2} k_1 \gg k_1$$

Immediate disappearance smaller than a wavelength.

$$\frac{dn}{d\omega} > 0 \Rightarrow v_g < v_p \rightarrow \text{normal dispersion}$$

$$\frac{dn}{d\omega} < 0 \Rightarrow v_g > v_p \rightarrow \text{anomalous dispersion}$$

Recall

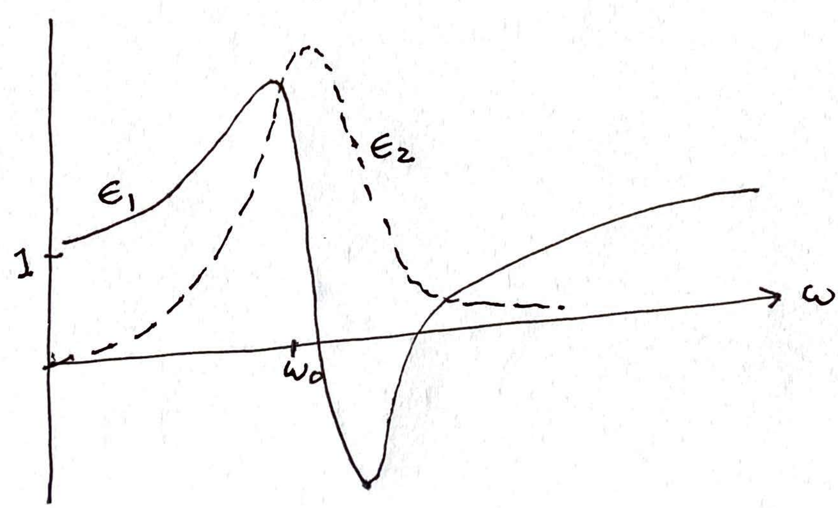
$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{\omega_p^2}{-\omega^2 + \omega_0^2 - i\gamma\omega}$$

So

$$\frac{\epsilon_1(\omega)}{\epsilon_0} = 1 - \frac{\omega_p^2(\omega^2 - \omega_0^2)}{(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2}$$

$$\frac{\epsilon_2(\omega)}{\epsilon_0} = \omega_p^2 \frac{\gamma\omega}{(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2}$$

$$\frac{\epsilon_{1,2}(\omega)}{\epsilon_0}$$



Kramers-Kronig Relations

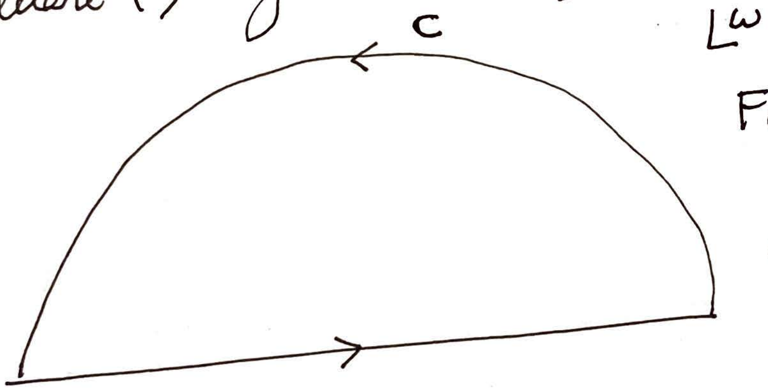
Recall the susceptibility χ

$$\chi(\omega) = \int_{-\infty}^{\infty} dt\, \chi(t)\, e^{i\omega t} \qquad (1)$$

$$\chi(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \chi(\omega)\, e^{-i\omega t}. \qquad (2).$$

and $\boxed{\chi(t<0) = 0}$

Evaluate (2) for $t<0$ by contour integration



For $t<0$
$e^{-i\omega t} \longrightarrow 0$
as $\omega \to \infty$
in upper-half plane

So $$\chi(t) = 2\pi i \sum_{\text{poles } \omega_i} \text{Residue}(\omega_i) \quad \text{for } t<0$$

$$= 0$$

$\Rightarrow$ There are no poles in UHP

$\Rightarrow$ $\underline{\chi(\omega) \text{ is analytic in UHP}}$.

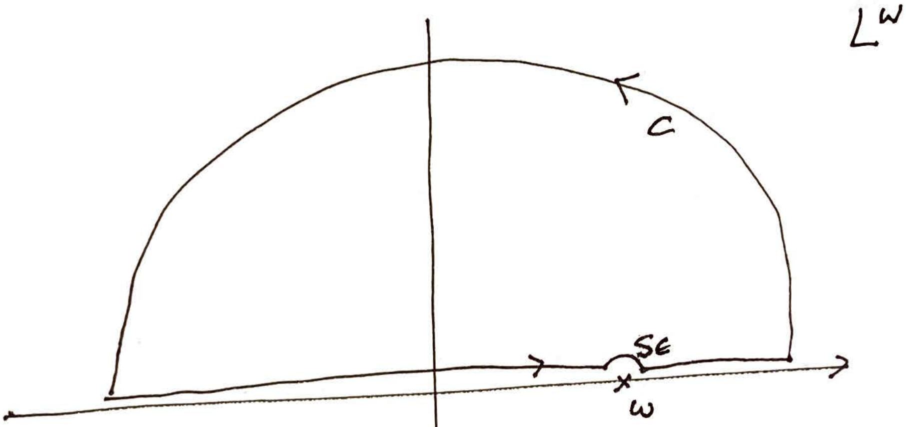
For $\chi(\omega) = \dfrac{\omega_p^2}{-\omega^2 + \omega_0^2 - i\gamma\omega}$

Poles at $\omega = -\dfrac{i\gamma}{2} \pm \sqrt{\omega_0^2 - \gamma^2/4}$ ✓ OK.

Recall from complex analysis

$$P\int dz \frac{f(z)}{z-z_0} = \int^{z_0-\epsilon} dz \frac{f(z)}{z-z_0} + \int_{z_0+\epsilon} dz \frac{f(z)}{z-z_0}$$

is the principal value as $\epsilon \to 0$.



Consider the integral

$$\frac{1}{i\pi}\oint_C d\omega' \frac{\chi(\omega')}{\omega'-\omega} = 0$$

$$\cancel{= \frac{1}{i\pi}\int_{-\infty}^{\infty} d\omega' \chi} = \frac{1}{i\pi} P\int_{-\infty}^{\infty} d\omega' \frac{\chi(\omega')}{\omega'-\omega}$$

$$\frac{1}{i\pi}\int_{S_\epsilon} d\omega' \frac{\chi(\omega')}{\omega'-\omega}$$

Now $$\int_{S_\epsilon} \frac{d\omega' \, X(\omega')}{\omega' - \omega} = -\frac{1}{2} 2\pi i \, \text{Res} \left[\frac{X(\omega')}{\omega' - \omega} \right]_{\omega' = \omega}$$

$$= -i\pi X(\omega). \qquad (*)$$

$\therefore$ $$\boxed{X(\omega) = \frac{1}{i\pi} P \int_{-\infty}^{\infty} \frac{d\omega' \, X(\omega')}{\omega' - \omega}}$$

or $$\text{Re}\, X(\omega) = P \int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \frac{\text{Im}\, X(\omega')}{\omega' - \omega}$$

$$\text{Im}\, X(\omega) = -P \int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \frac{\text{Re}\, X(\omega')}{\omega' - \omega}.$$

Alternative intepreletion of $(*)$

$$\boxed{\frac{1}{\omega \pm i\epsilon} = P\left(\frac{1}{\omega}\right) \mp i\pi \delta(\omega)}$$

$$\frac{1}{\omega \pm i\epsilon} = \frac{\omega}{\omega^2 + \epsilon^2} \mp i \frac{\epsilon}{\omega^2 + \epsilon^2}$$

$$= P\left(\frac{1}{\omega}\right) \mp i\pi \delta(\omega) \quad \text{as } \epsilon \to 0.$$