EM waves in metals

In the bulk of a metal ~~net static charge density = 0 (…)~~

~~However~~ $\vec{J}_{free} \neq 0$.

Drude model of free electrons

Electrons accelerate in applied electric field, and scatter off each other and impurities

$$m \frac{d\vec{v}}{dt} = q\vec{E} - \frac{m}{\tau}\vec{v}$$

(Just like model of an insulator but with $\omega_0 = 0$)

So if $\vec{E} = \vec{E}(\omega)\, e^{-i\omega t}$

Then $\left(-i\omega + \frac{1}{\tau}\right)\vec{v}(\omega) = \frac{q\vec{E}}{m}$

So

$$\vec{J}_{free} = nq\vec{V}$$

($n \to$ density of free electrons; ions do not move).

This gives ~~us~~

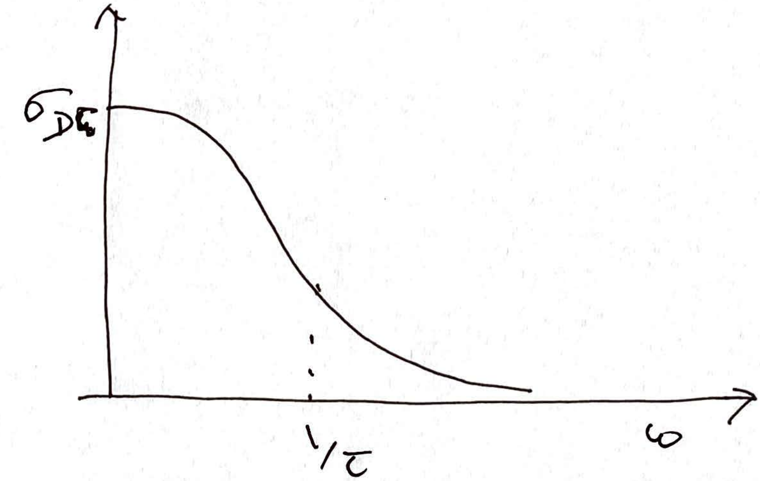
$$\vec{J}(\omega) = \sigma(\omega)\,\vec{E}(\omega)$$

where

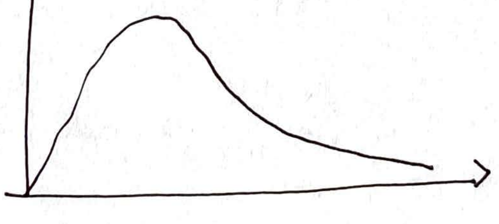
$$\sigma(\omega) = \frac{\sigma_{DC}}{1 - i\omega\tau}$$

$$\sigma_{DC} = \frac{ne^2\tau}{m} \quad \text{(Drude formula)}.$$

$\mathrm{Re}\,\sigma(\omega) = \sigma_1$



$\mathrm{Im}(\sigma) = \sigma_2$



Now we go back to Maxwell's equations

$$\nabla\cdot D = \rho \qquad \nabla\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$$

$$\nabla\cdot B = 0 \qquad \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

Constitutive relations

$$\vec{B} = \mu\vec{H}$$

$$\vec{D} = \epsilon(\omega)\vec{E}$$

$$\vec{J} = \sigma(\omega)\vec{E}$$

and from $\dfrac{\partial\rho}{\partial t} + \nabla\cdot J = 0$

$$\rho = \frac{\vec{k}\cdot\vec{J}}{\omega} = \frac{\sigma(\omega)}{\omega}\vec{k}\cdot\vec{E}$$

Now Maxwell's equtions become

$$i\left(\epsilon(\omega) + i\frac{\sigma(\omega)}{\omega}\right)\vec{k}\cdot\vec{E} = 0$$

$$\vec{k}\cdot\vec{B} = 0$$

$$i\vec{k}\times\vec{B} = -i\mu\omega\left(\epsilon(\omega) + \frac{i\sigma(\omega)}{\omega}\right)\vec{E}(\omega)$$

$$\vec{k}\times\vec{E} = \omega\vec{B}(\omega)$$

Just like in an insulator but with

$$\epsilon_{eff}(\omega) = \underset{\substack{\downarrow \\ \text{bound electrons}}}{\epsilon(\omega)} + i\underset{\substack{\downarrow \\ \text{free electrons}}}{\frac{\sigma(\omega)}{\omega}}$$

<u>Low frequencies</u>

$$\epsilon_{eff}(\omega) = \epsilon_1 + i\left(\epsilon_2 + \frac{\sigma_{DC}}{\omega}\right)$$

So $\epsilon_2 \gg \epsilon_1 > 0$ for $\omega > 0$.

~~This is the regime~~

This is the regime of <u>resonant absorption</u>

$$k = k_1 + ik_2 \approx \sqrt{i\mu\omega\sigma_{DC}}$$

$$\approx \sqrt{\frac{\mu\omega\sigma_{DC}}{2}}(1+i).$$

So the transverse EM wave decays in a <u>skin depth</u>

$$\delta = \frac{1}{k_2} = \sqrt{\frac{2}{\mu\omega\sigma_{DC}}}$$

High frequencies

For $\omega \gg 1/\tau$ $\quad \sigma(\omega) \simeq \dfrac{i\,\sigma_{DC}}{\omega\tau}$

and $\epsilon(\omega) \approx \epsilon_0$

$$\epsilon_{eff} \simeq \epsilon_0 - \frac{\sigma_{DC}}{\omega^2 \tau}$$

$$= \epsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2}\right)$$

So we always have $|\epsilon_{1\,eff}| > |\epsilon_{2\,eff}|$

Two cases

- $\epsilon_1 < 0$, for $\omega < \omega_p$

  Total ~~refletion~~ reflection

- $\epsilon_1 > 0$ for $\omega > \omega_p$

  Transparent!

ω_p ~~[illegible]~~ for most metals in the UV.

In the ionosphere, $\omega_p \simeq 2\pi\ 9\,\mathrm{MHz}$

transparent to FM radio

but not to AM radio.

Dispersion in metals and insulators

(Ignoring damping).

(Transverse waves)

We have

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2}$$

with $\omega_0 = 0$ in a metal.

and the relation

$$\boxed{k^2 = \omega^2 \mu_0 \epsilon(\omega).}$$

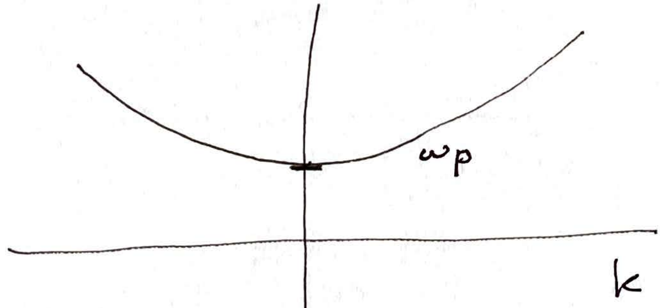
We will solve this eqn for $\omega(k)$.

(i) Metal

$$k^2 = \omega^2 \mu_0 \epsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2}\right)$$

$$= \frac{\omega^2}{c^2} - \frac{\omega_p^2}{c^2}$$

or $\omega = \sqrt{c^2 k^2 + \omega_p^2}$



No transverse waves for $\omega < \omega_p$.

(ii) Insulator

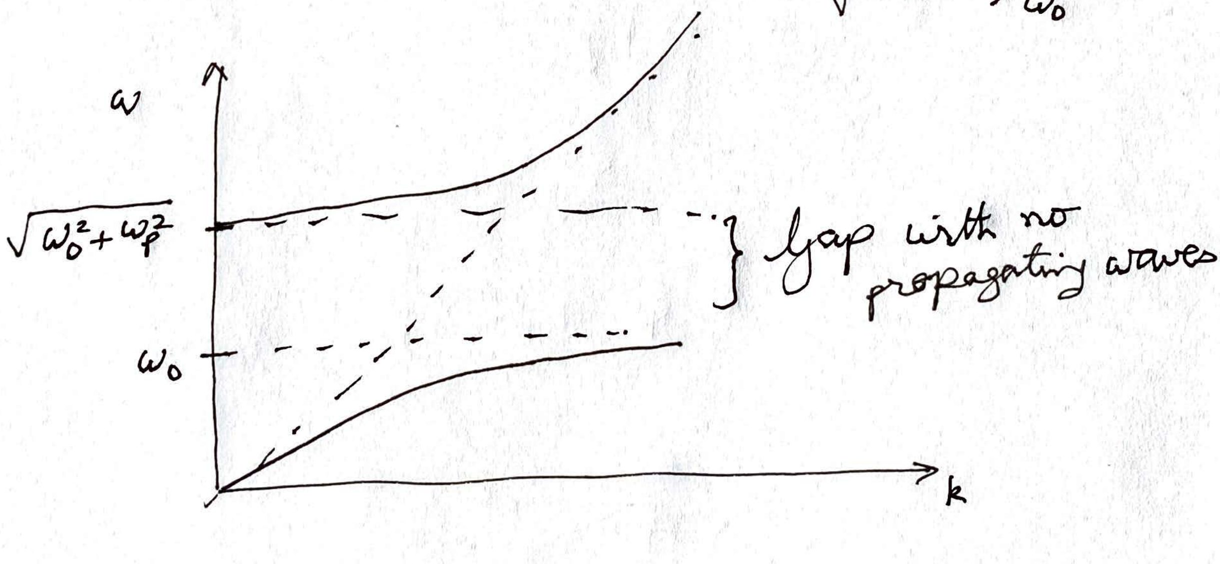
$$k^2 = \frac{\omega^2}{c^2}\left(1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2}\right)$$

~~Sove~~ Solve quadratic equation for ω

$$\omega^4 - (c^2k^2 + \omega_p^2 + \omega_0^2)\,\omega^2 + \omega_0^2 k^2 c^2 = 0$$

As $k \to \infty$ $\quad \omega = \begin{cases} ck \\ \omega_0 \end{cases}$

As $k \to 0$ $\quad \omega = \begin{cases} \sqrt{\omega_0^2 + \omega_p^2} \\ \dfrac{ck}{\sqrt{1 + \omega_p^2/\omega_0^2}} \end{cases}$



Longitudinal Waves — Plasma oscillations

So far we have considered only transverse waves with $\vec{k}\cdot\vec{E} = 0 = \vec{k}\cdot B$.

The ~~[illegible]~~ actual Maxwell Eqn is

$$\nabla\cdot D = \rho = i\left(\epsilon(\omega) + i\frac{\sigma(\omega)}{\omega}\right)\vec{k}\cdot\vec{E} = 0.$$

So there can be longitudinal wave with $\vec{k}\cdot\vec{E} \neq 0$ provided

$$\boxed{\epsilon(\omega) + i\frac{\sigma(\omega)}{\omega} = 0.}$$

or $$\epsilon_0 + i\,\cancel{[illegible]}\,\frac{ne^2\tau}{m\omega(1 - i\omega\tau)} = 0.$$

Consider

(i) $\omega\tau \ll 1$

Then $$\omega = -i\frac{\sigma_{DC}}{\epsilon_0}$$

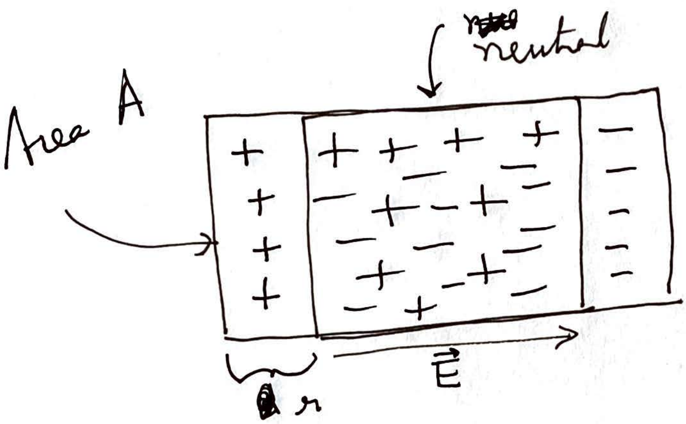
Decay of electric field and charge density in a time $\approx \dfrac{\epsilon_0}{\sigma_{DC}}$

(ii) $\omega\tau \gg 1$

Then

$$\omega^2 = \omega_p^2 = \frac{ne^2}{\epsilon_0 m}$$

Plasma frequency → independent of τ



Total charge in this region = $neA\Delta r$

Surface charge density = $ne\Delta r$

Electric field (Gauss's) law

$$= \frac{ne\Delta r}{\epsilon_0}$$

~~Mass of electrons in region~~

Equation of motion of an electron

$$m\frac{d^2\vec{r}}{dt^2} = -\frac{ne^2}{\epsilon_0}\vec{r}$$

S.H.O. at $\omega = \omega_p$!