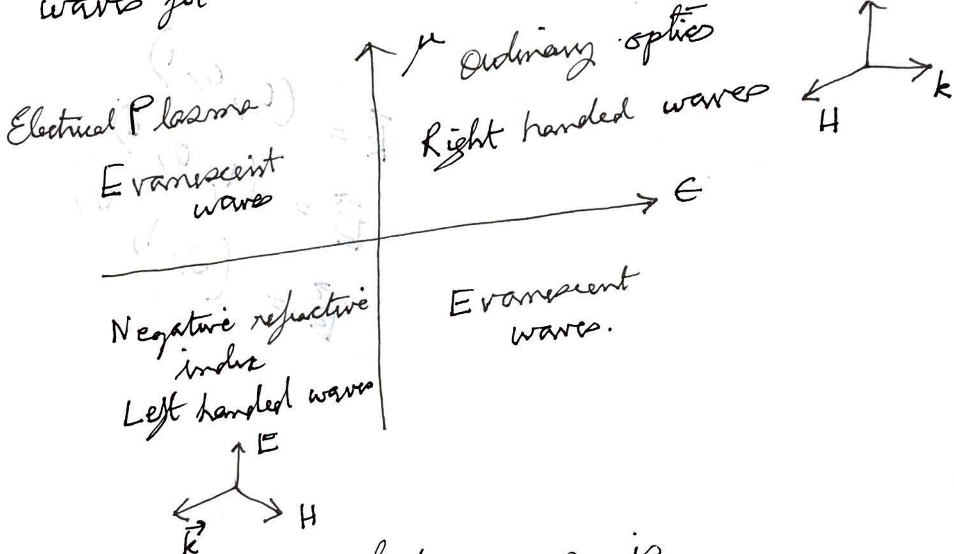
Negative index of refraction

Dispersion relation for EM waves

$$\boxed{k^2 = \mu(\omega)\,\epsilon(\omega)\,\omega^2}$$

Waves propagate for $\mu\epsilon > 0$.

In principle, we can also have waves for $\epsilon < 0$ and $\mu < 0$.



The velocity of the wave is

$$v = \frac{1}{\sqrt{\mu\epsilon}}.$$

and the index of refraction $n = \dfrac{c}{v}$

or $n^2 = \mu\epsilon/\mu_0\epsilon_0$

It turns out that

$$\boxed{\begin{aligned} n &= -\sqrt{\epsilon\mu}/\sqrt{\epsilon_0\mu_0} \quad \text{for } \epsilon<0,\ \mu<0 \\ n &= \sqrt{\epsilon\mu}/\sqrt{\epsilon_0\mu_0} \quad \text{for } \epsilon>0,\ \mu>0. \end{aligned}}$$

Poynting vector for an EM wave is

$$\vec{S} = \vec{E}\times\vec{H} = \frac{1}{\mu}\vec{E}\times\vec{B}$$

$\frac{1}{\mu} \longrightarrow$ note change of sign!

For an EM wave

$$\vec{E}(\vec{r},t) = \vec{E}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$\vec{B}(\vec{r},t) = \vec{B}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

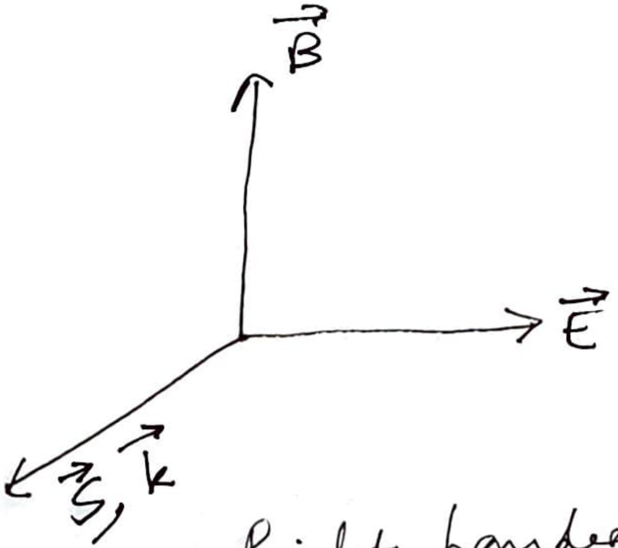
Maxwells equations

$$\boxed{\begin{aligned} &\vec{k}\cdot\vec{E} = 0 \qquad \vec{k}\cdot\vec{B} = 0 \\ &\vec{k}\times\vec{B} = -\mu\epsilon\omega\,\vec{E} \\ &\vec{k}\times\vec{E} = \omega\,\vec{B} \end{aligned}}$$

So $\vec{S} = \frac{1}{\mu}\vec{E}\times\vec{B} = \frac{\vec{E}\times(\vec{k}\times\vec{E}^*)\sqrt{\epsilon\mu}}{\mu|k|}$

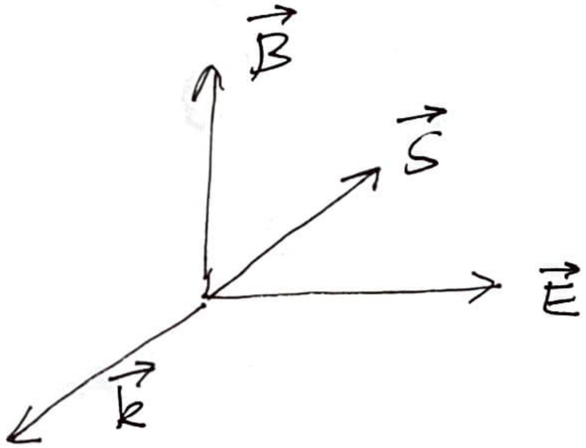
$$= \frac{\sqrt{\epsilon\mu}}{\mu}\frac{\vec{k}}{|k|}|\vec{E}|^2$$

So $\epsilon > 0, \mu > 0$



Right handed material

$\epsilon < 0, \mu < 0$



Left handed material.

Interface equations

$$\vec{E}_1,\ \vec{B}_1,\ \epsilon_1,\ \mu_1$$

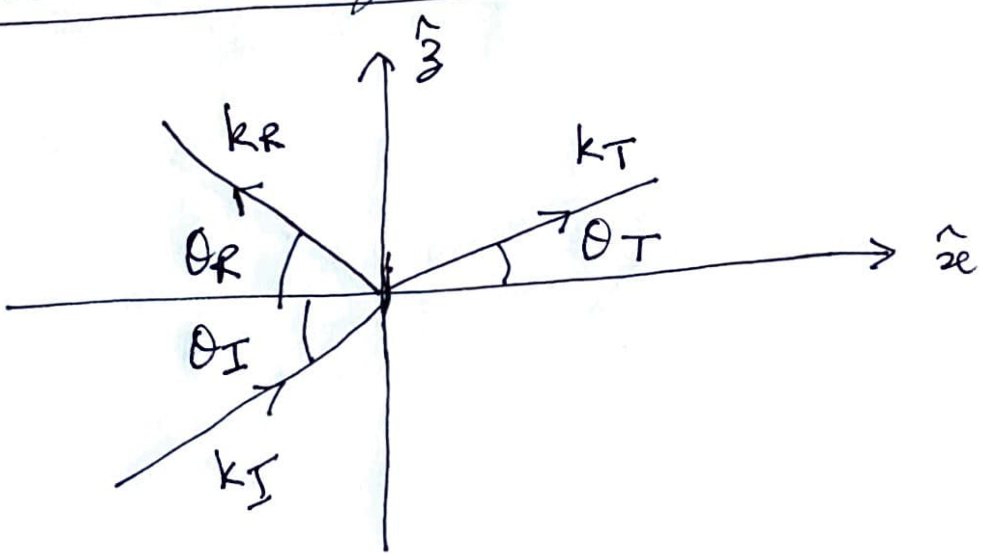
$\hat{n}\uparrow$

$$\vec{E}_2,\ \vec{B}_2,\ \epsilon_2,\ \mu_2$$

Boundary conditions

$$\boxed{\begin{aligned}
&\hat{n}\cdot(\vec{B}_1-\vec{B}_2)=0\\
&\hat{n}\cdot(\epsilon_1\vec{E}_1-\epsilon_2\vec{E}_2)=0\\
&\hat{n}\times(\vec{E}_1-\vec{E}_2)=0\\
&\hat{n}\times\left(\frac{\vec{B}_1}{\mu_1}-\frac{\vec{B}_2}{\mu_2}\right)=0
\end{aligned}}$$

Recall usual refraction



From matching boundary conditions we obtained

$$k_I = k_R = \omega\sqrt{\epsilon_1\mu_1}$$

$$k_T = \omega\sqrt{\epsilon_2\mu_2}$$

$$\vec{k}_I \cdot \hat{z} = \vec{k}_R \cdot \hat{z} = \vec{k}_T \cdot \hat{z}$$

$$\Rightarrow \quad k_I \sin\theta_I = k_R \sin\theta_R = k_T \sin\theta_T$$

Solution is $k_I = k_R$; $\theta_I = \theta_R$

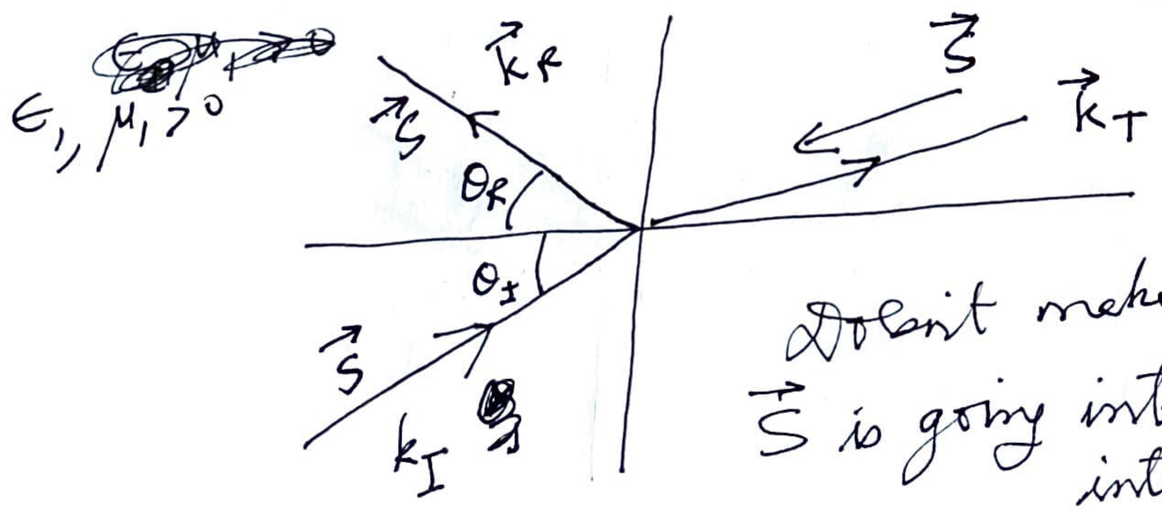
and $\omega\sqrt{\epsilon_1\mu_1}\,\sin\theta_I = \omega\sqrt{\epsilon_2\mu_2}\,\sin\theta_T$

Snell's law $\boxed{n_1 \sin\theta_I = n_2 \sin\theta_T}$

<u>Now consider right handed material</u>

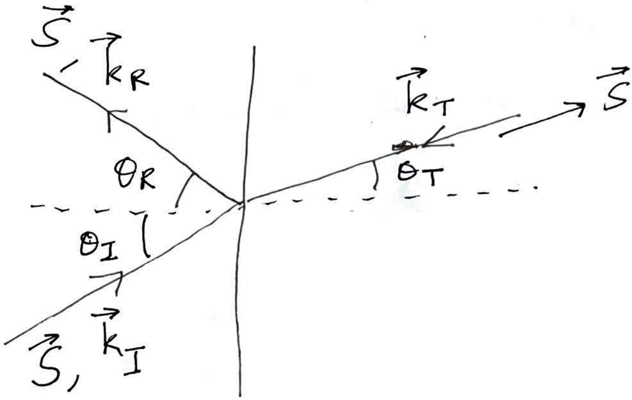
The previous configuration

$\epsilon_2, \mu_2 < 0$



Doesn't make sense!
$\vec{S}$ is going into interface

Energy must flow away from interface
(Can also obtain by matching boundary conditions)



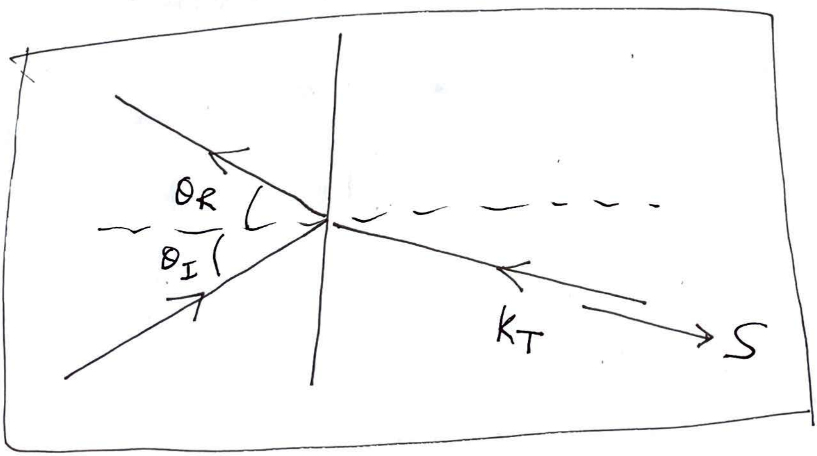
Matching $\vec{k}\cdot\vec{r}$ as before we now obtain

$$\sqrt{\epsilon_1\mu_1}\sin\theta_I = -\sqrt{\epsilon_2\mu_2}\sin\theta_T$$

i.e. $\theta_T < 0$!

$$n_2 = \frac{-\sqrt{\epsilon_2\mu_2}}{\sqrt{\epsilon_0\mu_0}}$$

Actual situation

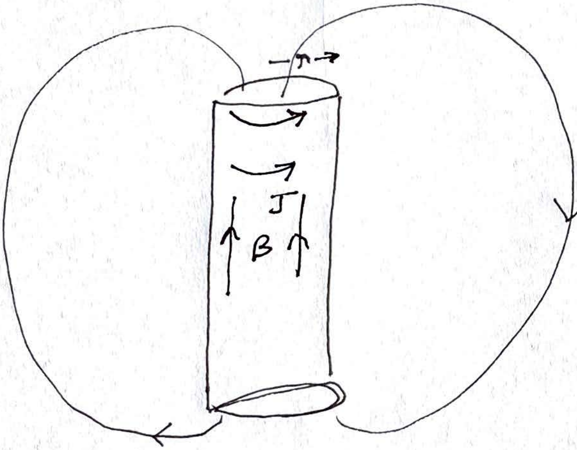


How do we obtain a $\mu < 0$?

Requires a nano-structured meta material (Pendry et al 1999)

Use a lattice of hollow metallic cylinders.

First consider a single metallic cylinder



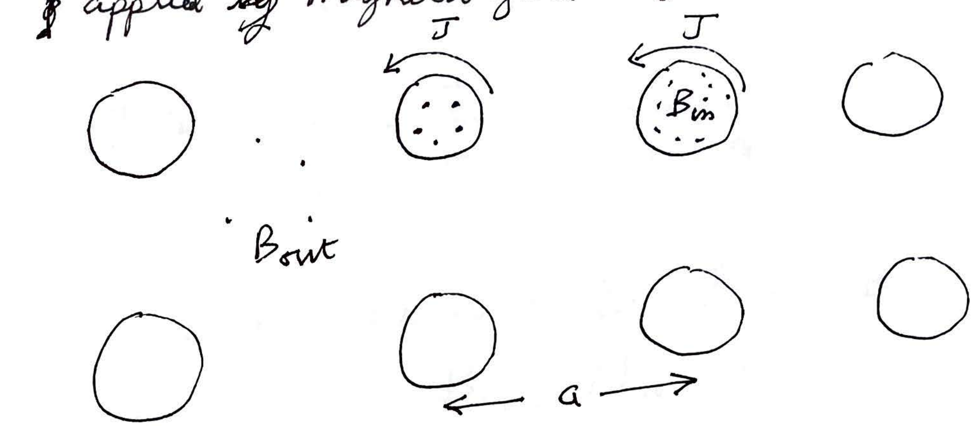
$J \rightarrow$ current per unit length

From Ampere's law $B = \mu_0 J$ inside cylinder.

$$\int_S B.dS = 0$$ on any infinite plane.

Flux inside = - Flux outside.

Now for ~~an~~ infinite lattice with an applied ~~B~~ magnetic field B_0



$$B_{in} = B_0 + \mu_0 J - \mu_0 \frac{\pi r^2}{a^2} \vec{J}$$

$$B_{out} = B_0 - \mu_0 \frac{\pi r^2}{a^2} \vec{J}$$

Correction from flux outside cylinders

Chosen so that

~~$\int B.dS = B_0 A$ independent of J.~~

$$\int_{\text{unit cell}} B.dS = B_0 a^2$$

unit cell independent of J

Now apply Faraday's law

$$\pi r^2 \frac{d}{dt}\left(B_0 + \mu_0 J - \frac{\pi r^2}{a^2}\mu_0 J\right) = -2\pi r \sigma J$$

where σ is the ~~conductivity~~ resistivity per unit area.

This yields for $B_0 \to B_0 e^{-i\omega t}$

$$J(\omega) = \frac{-B_0}{\mu_0\left(1 - \frac{\pi r^2}{a^2}\right) + i\,\frac{2\sigma}{\omega r}}$$

We want to characterize this $\vec{J}$ and $\vec{B}(r)$ by an effective B_{av}, M_{av}, and H_{av} per unit cell.

Q. How do we do the average ?

Integral form of Maxwells equation

$$\int_C H \cdot dl = \frac{\partial}{\partial t}\int D \cdot ds$$

$$\int_C E \cdot dl = -\frac{\partial}{\partial t}\int_S B \cdot ds$$

This implies we should average B over the unit cell <u>boundary</u>,

and average H over the unit cell <u>edges</u>.

Note: we are considering waves with $\lambda \gg a$, hence the averaging.

With this prescription we ~~obtain~~ obtain

$$\boxed{\begin{aligned} B_{av} &= B_0 \\ H_{av} &= \frac{1}{\mu_0} B_{outside} \end{aligned}}$$

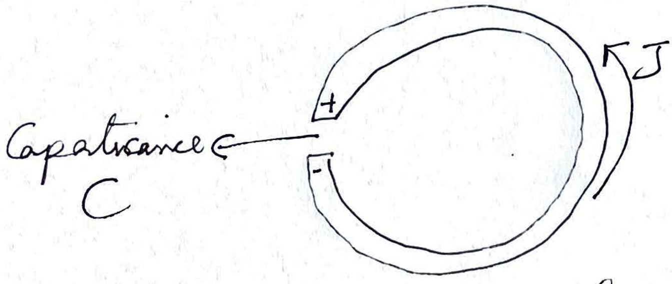
So $\mu_{av} = \dfrac{B_{av}}{H_{av}}$

$$\boxed{\mu_{av} = \mu_0 \left(1 - \frac{\pi r^2 / a^2}{1 + \dfrac{2 i \sigma}{\omega \mu_0 r}} \right)}$$

This not yet sufficient to yield a $Re[\mu_{av}] < 0$.

Solution: introduce a ~~of~~ capacitance to make it like a resonant circuit

Split-ring resonators



Then Faraday's law is modified to

$$\pi r^2 \left(B_0 + \mu_0 J - \mu_0 \frac{\pi r^2}{a^2} J\right)(-i\omega)$$

$$= -2\pi r \sigma J + \frac{J}{i\omega C}$$

new term

Now proceeding as before

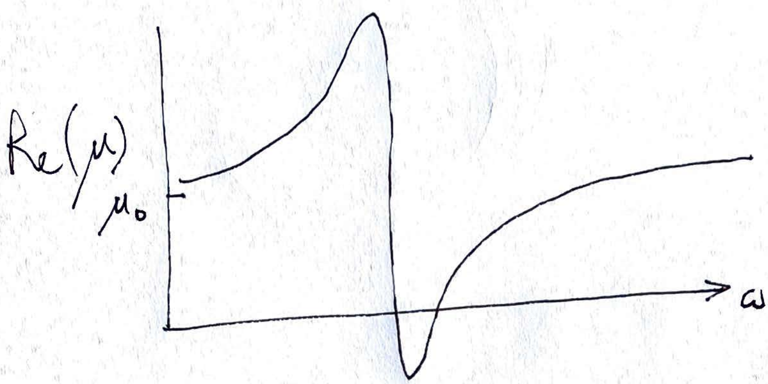
$$\mu = \mu_0 \left(1 - \frac{\pi r^2/a^2}{1 + \dfrac{2i\sigma}{\omega \mu_0 r} - \dfrac{1}{\omega^2 C \pi r^2 \mu_0}} \right)$$

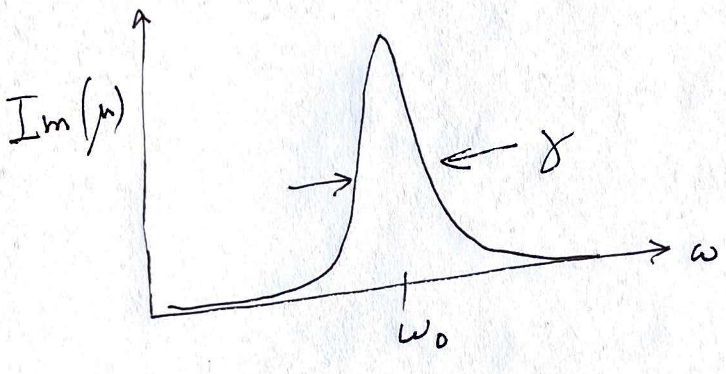
$$= \mu_0 \left(1 + \frac{f \omega^2}{\omega_0^2 - \omega^2 - i\gamma\omega} \right)$$

with $f = \pi r^2/a^2$, $\gamma = \dfrac{2\sigma}{\mu_0 r}$

$$\omega_0^2 = \frac{1}{C \pi r^2 \mu_0} \longrightarrow \frac{1}{LC}$$ with inductance $\pi r^2 \mu_0$.

$$\mu = \mu_0 \left(1 + \frac{f\,\omega^2}{\omega_0^2 - \omega^2 - i\gamma\omega} \right)$$





For $\gamma \ll \omega_0$ the resonance is sharp and μ can <u>be negative</u>.