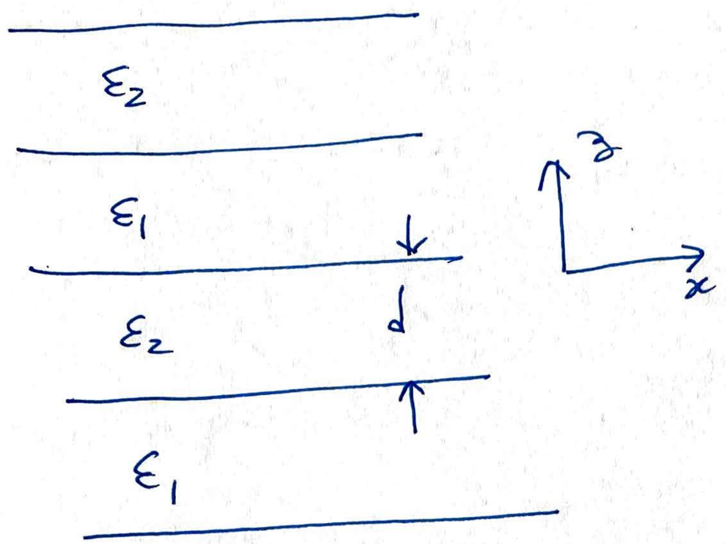
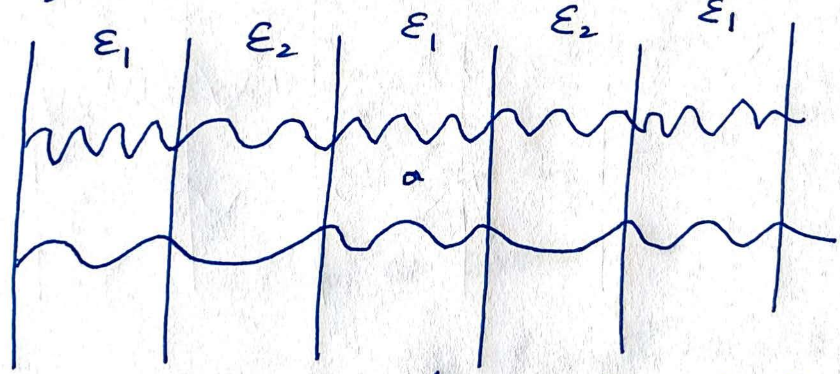
Photonic Lattices

We consider a periodic slab arrangement

μ equal



Interested in modes propagating in the z direction



$\varepsilon_1 > \varepsilon_2$

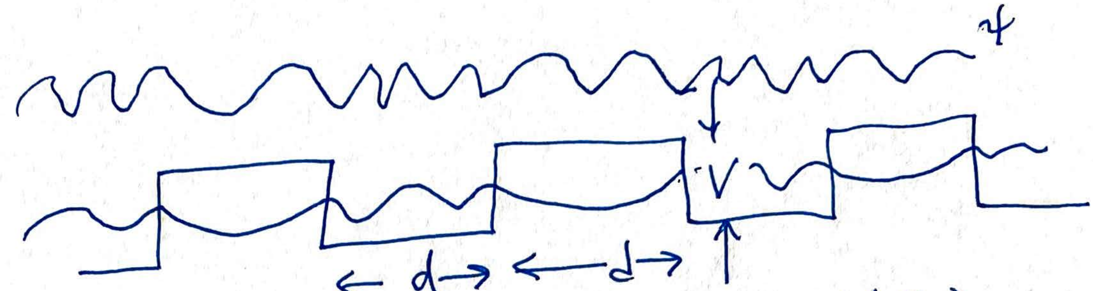
We will find that such waves can propagate but there are <u>band gaps</u>

No waves can propagate in certain frequency windows.

For TE modes the equations and boundary conditions are <u>identical</u> to a quantum problem in one dimension

So we consider a simple model of a one-dimension crystal.

$V(x)$



$V(x)$ is periodic $\quad V(x+2d) = V(x)$

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\psi(x) = E\psi(x). \quad (*)$$

<u>Bloch's Theorem</u>

Every solution of $(*)$ can be chosen to satisfy $\psi(x+2d) = e^{iK2d}\psi(x)$

(Also choose $-\pi < 2Kd < \pi$).

<u>Proof</u>: Consider the translation operator

$$\hat{T}f(x) = f(x+2d).$$

It is easy to show $[\hat{T}, \hat{H}] = 0$ where $\hat{H}$ is the Hamiltonian.

So $\hat{T}$ and $\hat{H}$ have a common set of eigenvectors.

~~Q~~ Eigenstates of $\hat{T}$

$$\hat{T} f(x) = \lambda f(x)$$

or $f(x+2d) = \lambda f(x)$.

We don't want exponentially increasing solutions. So must choose $|\lambda| = 1$.

We choose to write $\lambda = e^{2iKd}$ for convenience.

What is eigenfunction $f(x)$?

Claim $f(x) = \sum_n c_n e^{iG_n x} e^{iKx}$

where $G_n = \frac{2\pi n}{2d}$, n - integer (reciprocal lattice vectors).

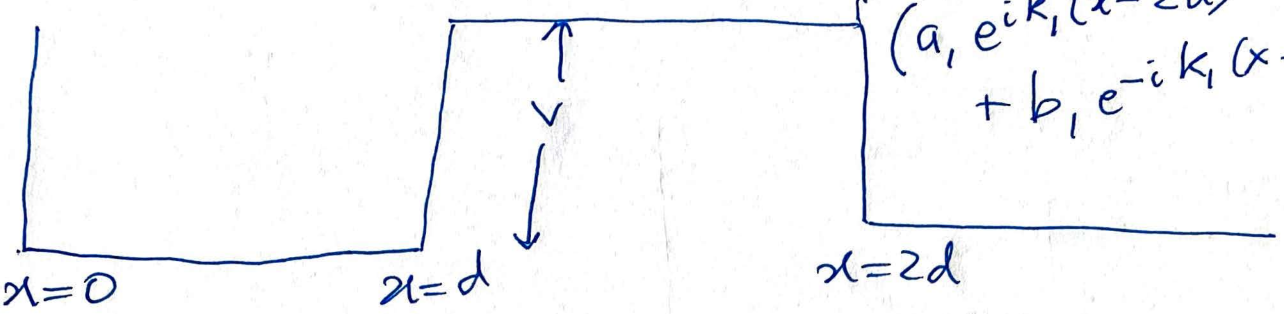
Easy to verify.

<u>Corollary</u>:- We can write the wavefunction

$$\boxed{\begin{array}{l} \psi(x) = e^{iKx} u(x) \\ \text{where } u(x+2d) = u(x) \end{array}}$$

$K \to$ crystal momentum or Bloch wavevector.

$a_1 e^{ik_1 x} + b_1 e^{-ik_1 x}$ | $a_2 e^{ik_2 x} + b_2 e^{-ik_2 x}$ | $\cancel{e^{ik(x-2d)}}$ $e^{-i2Kd} \otimes \left(a_1 e^{ik_1(x-2d)} + b_1 e^{-ik_1(x-2d)}\right)$.



$$k_1 = \frac{\sqrt{2Em}}{\hbar} \qquad k_2 = \frac{\sqrt{2m(E-V)}}{\hbar}$$

(could be complex)

Matching boundary conditions at $x = d$.

Continuity of $\psi(x)$

$$a_1 e^{ik_1 d} + b_1 e^{-ik_1 d} = a_2 e^{ik_2 d} + b_2 e^{-ik_2 d}$$

Continuity of $\psi'(x)$

$$k_1\left(a_1 e^{ik_1 d} - b_1 e^{-ik_1 d}\right) = k_2\left(a_2 e^{ik_2 d} - b_2 e^{-ik_2 d}\right).$$

Boundary conditions at $x = 2d$

$$a_2 e^{ik_2 2d} + b_2 e^{-ik_2 2d} = e^{-i2Kd}(a_1 + b_1)$$

$$k_2\left(a_2 e^{ik_2 2d} - b_2 e^{-ik_2 2d}\right) = k_1 e^{-2iKd}(a_1 - b_1).$$

4 homogeneous equations $\rightarrow$ determinant should vanish

This leads to

$$\boxed{\cos(2Kd) = \cos(k_1 d)\cos(k_2 d) - \frac{k_1^2 + k_2^2}{2k_1 k_2}\sin(k_1 d)\sin(k_2 d)}$$

~~Note~~ Recall

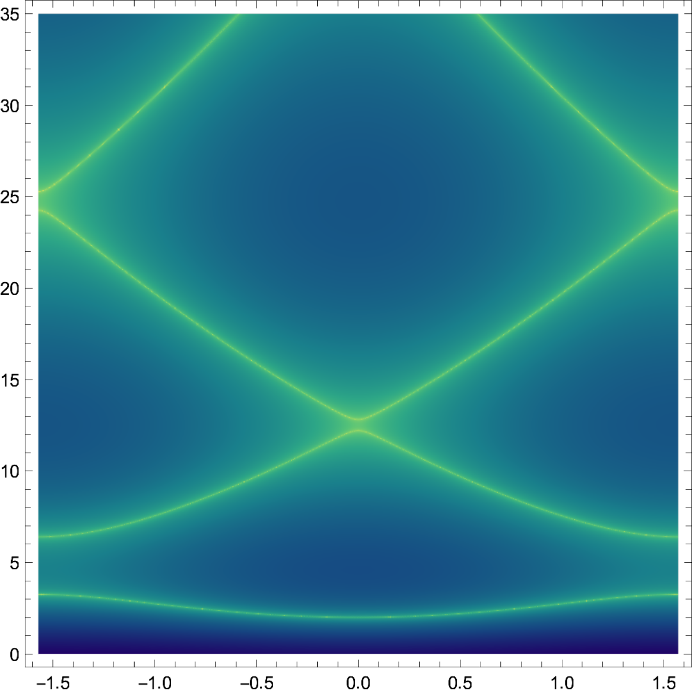
$$k_1 = \frac{\sqrt{2mE}}{\hbar}$$

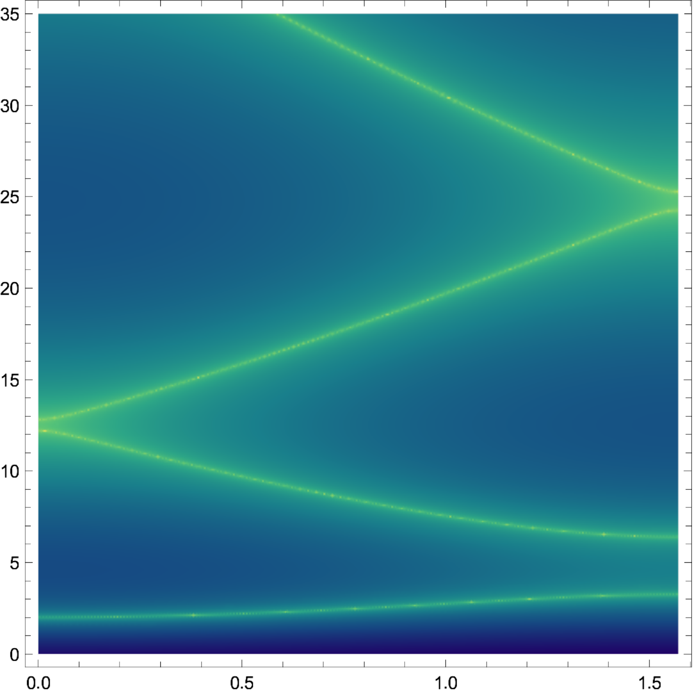
$$k_2 = \frac{\sqrt{2m(E-V)}}{\hbar}$$

Solve eqns for E as a function of K.

Note: solutions with $E < V$, k_2 imaginary can appear.

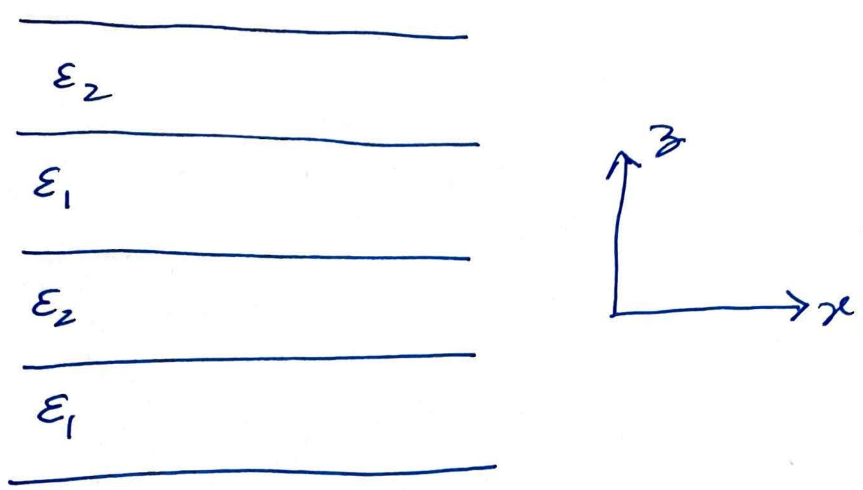
Plot of band structure E(K) for V=5, d=1


35
30
25
20
15
10
5
0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5



35
30
25
20
15
10
5
0
0.0
0.5
1.0
1.5

Return to Optics



TE mode:

$$\vec{E}(\vec{r}) = e^{i(k_x x + k_y y)}\, \hat{n}_x\, E(z)$$

Equations for $E(z)$ are identical to those of the quantum wavefunction $\psi(x)$ with

$$\underset{\text{quantum}}{2m(E - V(x))/\hbar^2} \xrightarrow{\text{mapping to}} \begin{cases} \dfrac{\omega^2}{c^2} n_1^2 - k_x^2 - k_y^2 \equiv \gamma_1^2 \\[2ex] \dfrac{\omega^2}{c^2} n_2^2 - k_x^2 - k_y^2 \equiv \gamma_2^2 \end{cases}$$

Then the solution for the crystal momentum is

$$\cos(2Kd) = \cos(\gamma_1 d)\cos(\gamma_2 d) - \frac{1}{2}\left(\frac{\gamma_1^2 + \gamma_2^2}{\gamma_1 \gamma_2}\right) \sin(\gamma_1 d)\sin(\gamma_2 d)$$

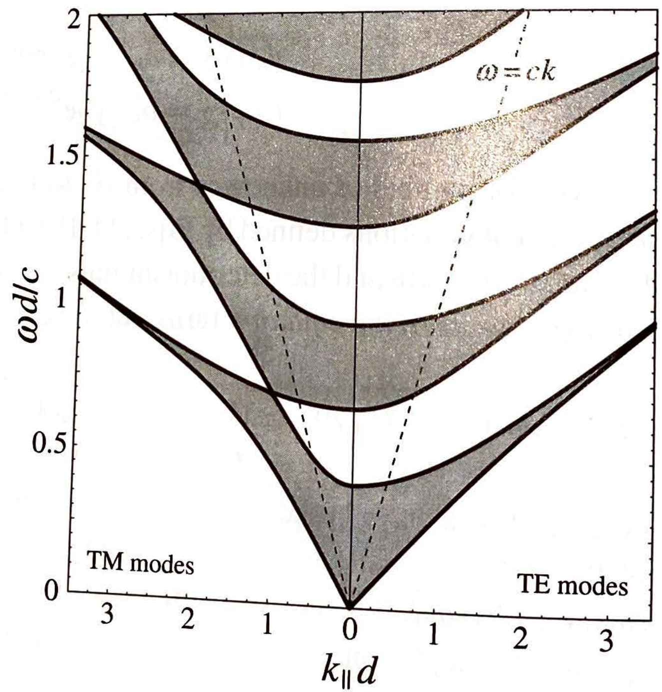


Fig. 11.3 The band diagram for a one-dimensional photonic crystal. The shaded areas are the allowed bands. The diagram represents both TE and TM modes. For a one-dimensional photonic crystal, there are no complete bandgaps, i.e. t are no frequencies for which propagation is inhibited in all directions. Values used: $\varepsilon_1 = 2.33$ (SiO_2) and $\varepsilon_2 = 17$ (InSb).