Metallic Waveguides

Free charges in a metal move in response to an applied field. These move to a surface layer so that there is no electric field in the bulk of the metal. ~~Similar~~ comments apply to time-dependent magnetic fields.

So we have surface charge density Σ and a surface current density $\vec{K}$ so that

$$\vec{n} \cdot \vec{D} = \Sigma \quad \text{and} \quad \vec{n} \times \vec{H} = \vec{K}$$

where $\vec{D}$ and $\vec{H}$ are the fields outside the conductor.

From the other Maxwell's equations we have

$$\boxed{\vec{n} \times \vec{E} = 0 \qquad \vec{n} \cdot \vec{B} = 0}$$

Electric fields are tangential, while time-dependent magnetic fields are normal to the surface in a perfect conductor. In a real metal, the fields decay a distance of order the skin depth

i.e. $\delta = \sqrt{\dfrac{2}{\mu \omega \sigma}}$

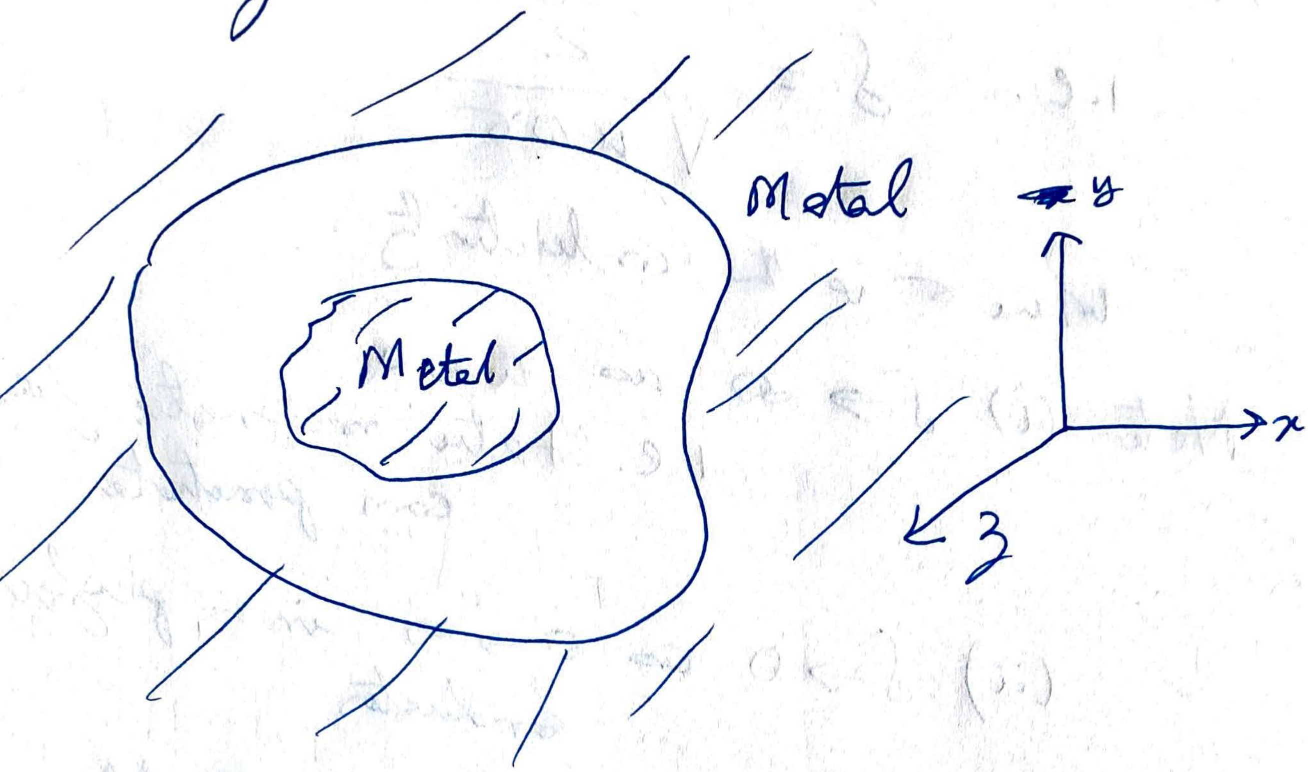
where σ is the conductivity.

Note (i) $\delta \to \infty$ as $\omega \to 0$
i.e. static magnetic fields can penetrate

(ii) $\delta \to 0$ as $\sigma \to \infty$ in a perfect conductor

(iii) This is distinct from the Meissner effect which expels static magnetic fields in a superconductor.

Guided waves



In the cavity region

$$\vec{E}(x,y,z,t) = \vec{E}(x,y)\, e^{i(kz-\omega t)}$$

$$\vec{B}(x,y,z,t) = \vec{B}(x,y)\, e^{i(kz-\omega t)}$$

We also write

$$\vec{E} = \vec{E}_T + E_z \hat{z}$$

$$\vec{B} = \vec{B}_T + B_z \hat{z}$$

with $\hat{z}\cdot\vec{E}_T = \hat{z}\cdot\vec{B}_T = 0$

and denote gradient in the xy plane by $\vec{\nabla}_T$

Then Maxwell's equations are

$$ik\vec{E}_T \cancel{\frac{\partial \vec{E}_T}{\partial z}} + i\omega\hat{z}\times\vec{B}_T = \vec{\nabla}_T E_z$$

$$\hat{z}\cdot(\vec{\nabla}_T \times E_T) = i\omega B_z$$

$$ik\vec{B}_T \cancel{\frac{\partial \vec{B}_T}{\partial z}} - i\mu\epsilon\omega\,\hat{z}\times E_T = \vec{\nabla}_T B_z$$

$$\hat{z}\cdot(\vec{\nabla}_T \times \vec{B}_T) = -i\mu\epsilon\omega E_z$$

$$\vec{\nabla}_T\cdot\vec{E}_T = -\frac{\partial E_z}{\partial z}$$

$$\vec{\nabla}_T\cdot\vec{B}_T = -\frac{\partial B_z}{\partial z}$$

These equations imply the Helmholtz equation for all component

$$\left[\nabla_T^2 + (\mu\epsilon\omega^2 - k^2)\right]\begin{Bmatrix}\vec{E}_T\\ \vec{B}_T\\ E_z\\ B_z\end{Bmatrix} = 0$$

We have combine these equations with boundary conditions $\hat{n}.\vec{B} = \hat{n}\times\vec{E} = 0$

(A) TEM modes

Are there any solutions with

$$E_z = B_z = 0 \quad ?$$

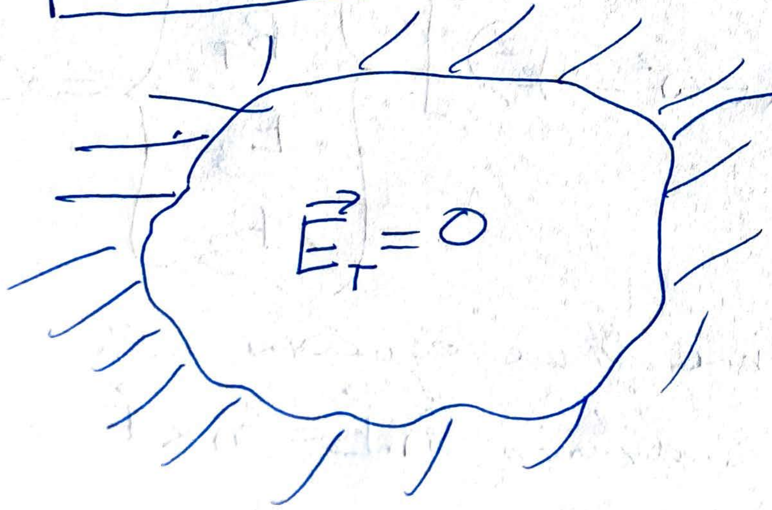
Equations reduce to

$$\boxed{\begin{gathered}\hat{z}\cdot(\vec{\nabla}_T \times \vec{E}_T) = \vec{\nabla}_T \cdot \vec{E}_T = 0 \\ \hat{n} \times \vec{E}_T = 0 \\ \hat{n} \cdot E_T = \Sigma\end{gathered}} \rightarrow \text{Electrostatics in 2D!}$$

$$\boxed{\begin{gathered}\hat{z}\cdot(\vec{\nabla}_T \times \vec{B}_T) = \vec{\nabla}_T \cdot \vec{B}_T = 0 \\ \hat{n} \cdot \vec{B}_T = 0 \\ \hat{n} \times \vec{B}_T = K\end{gathered}} \rightarrow \text{Magnetostatics in 2D!}$$

AND

$$\boxed{\sqrt{\mu\epsilon}\,\omega = k} \rightarrow \text{linear dispersion!}$$



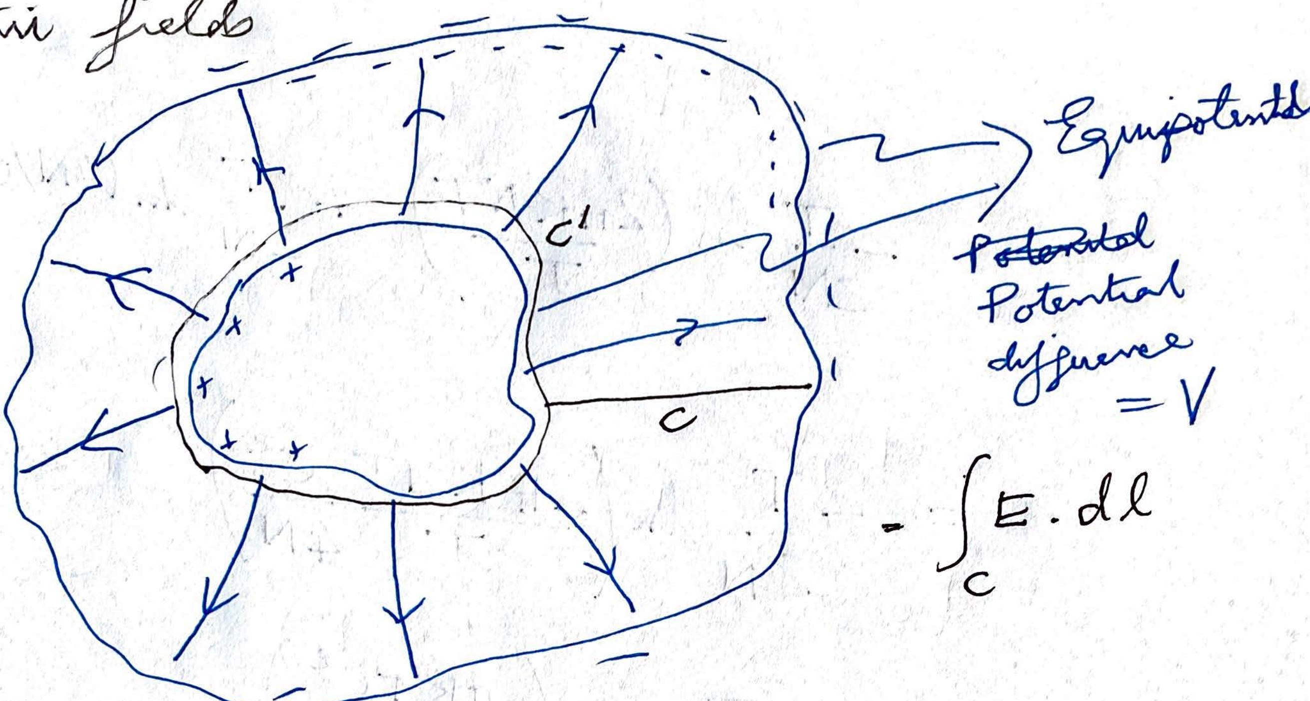
→ Equipotential

$\vec{E}_T = 0$

NOT POSSIBLE IN A HOLLOW WAVEGUIDE!

Coaxial cable

Electric fields



Let $Q = \oint_{C'} \Sigma \, dl$ be the charge per unit length.

Then by solving ~~electrostatics~~ electrostatics

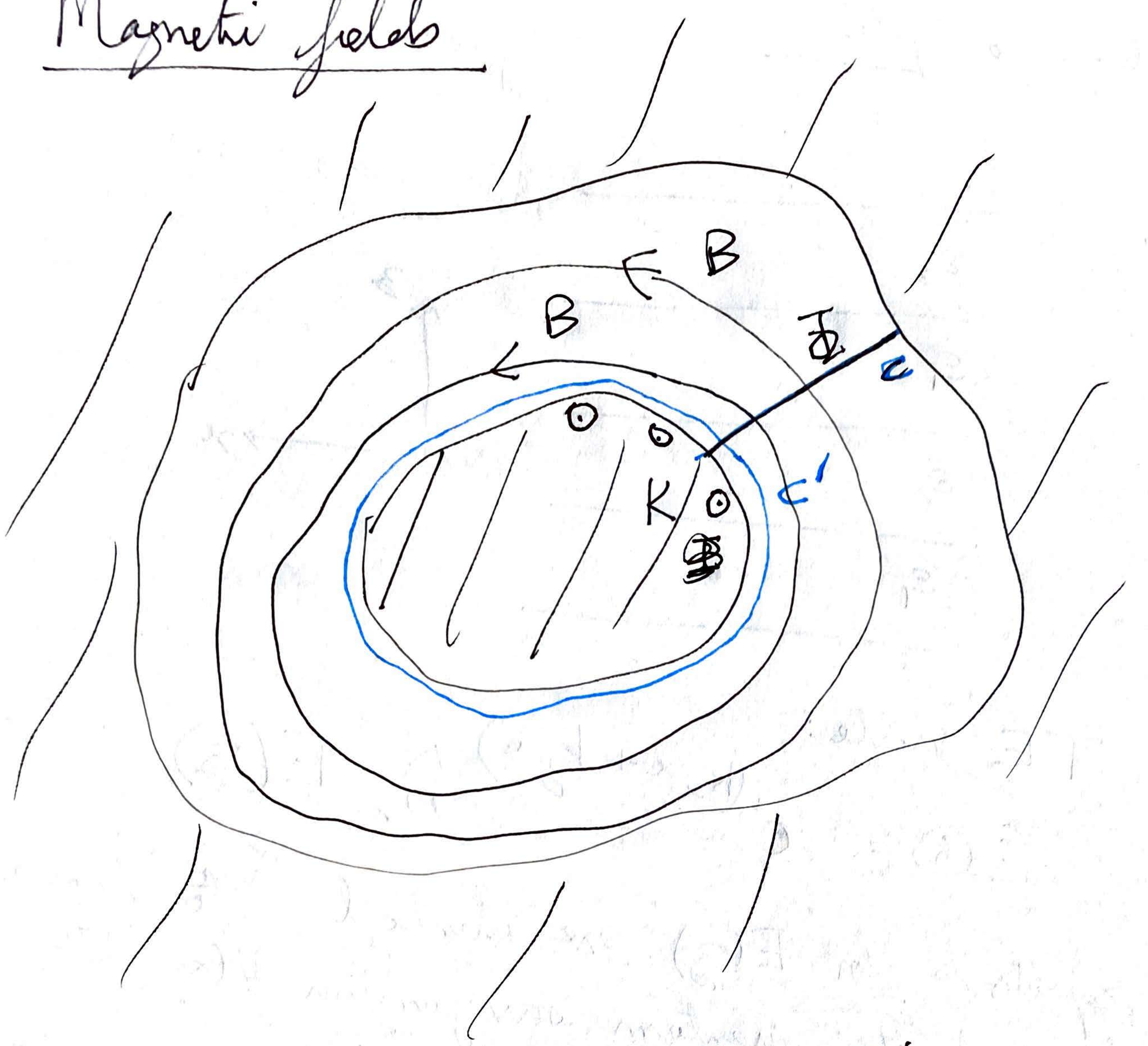
$$\vec{E} = -\nabla\phi$$

$$\nabla^2\phi = 0$$

we can determine the capacitance per unit length via

$$Q = CV$$

Magnetic fields



Now we write $B = -\nabla \psi$

Then $\nabla^2 \psi = 0$.

In fact $f = \phi + i\psi$ is an analytic function of $x + iy$, and (ϕ, ψ) obey the Cauchy-Riemann equations

We define the flux $\Phi = \int_C dl \cdot B$

and the current $\int_{C'} K \, dl = I$.

Then these obey

$$\Phi = L I$$

where L is the inductance per unit length.

In fact, the relationship between ϕ and ψ

implies

$$\boxed{LC = \mu\epsilon}$$

By integrating Maxwell's equations along C

and C'

$$ik\vec{E}_T + i\omega\hat{z}\times\vec{B}_T = 0$$

$$ik\vec{B}_T - i\mu\epsilon\omega(\hat{z}\times\vec{E}_T) = 0$$

we obtain

$$ikV = i\omega L I$$

and

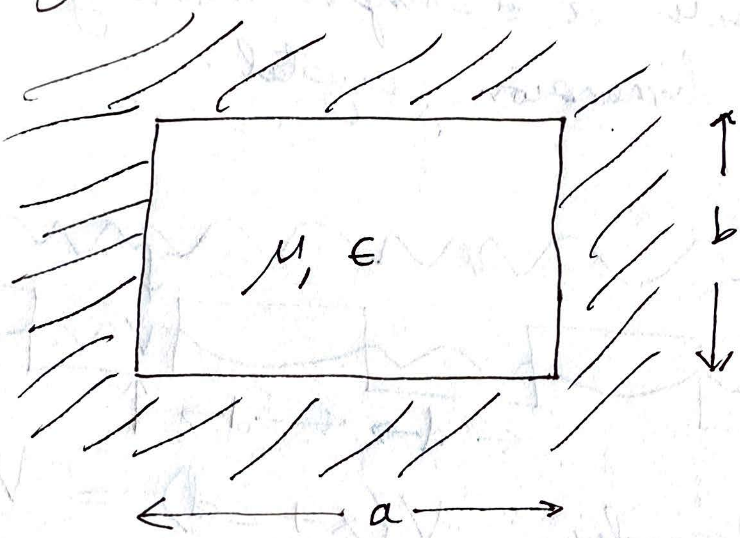
$$ikI = i\omega C V$$

or

$$\boxed{\begin{aligned}\frac{\partial V}{\partial z} &= -L\frac{\partial I}{\partial t}\\ \frac{\partial I}{\partial z} &= -C\frac{\partial V}{\partial t}\end{aligned}}$$

Telegraph equations.

Rectangular Waveguides



TE mode

$$E_z = 0$$

$$ik E_T + i\omega\, \hat{z} \times \vec{B}_T = 0$$

$$\vec{\nabla}_T \times \vec{B}_T = 0$$

$$\hat{z} \cdot (\vec{\nabla}_T \times \vec{E}_T) = i\omega B_z$$

$$ik B_T - i\mu\epsilon\omega\, (\hat{z} \times \vec{E}_T) = \vec{\nabla}_T B_z$$

These equations reduce to
with $B_z = \psi$

$$\boxed{(\nabla_T^2 + \gamma^2)\psi = 0 \; ; \; \gamma^2 = \mu\epsilon\omega^2 - k^2}$$

$$\boxed{\vec{B}_T = \frac{ik}{\gamma^2} \vec{\nabla}_T \psi \; ; \; \vec{E}_T = -\frac{\omega}{k} \hat{z} \times \vec{B}_T}$$

Soln of $(\nabla_T^2 + \gamma^2)\psi = 0$

with $n \cdot \vec{B}_T = 0$

$$\psi_{mn}(x, y) = A \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$$

$$\gamma_{mn}^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

$m = 0, 1, 2, 3 \ldots$
$n = 0, 1, 2, 3 \ldots$
$mn \neq 0$.

So the dispersion of the mn mode is

$$\omega^2 = \sqrt{\frac{k^2}{\mu\epsilon} + \omega_{mn}^2}$$

where

$$\omega_{mn} = \frac{\pi}{\sqrt{\mu\epsilon}} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^{1/2}$$

are the cutoff frequencies.