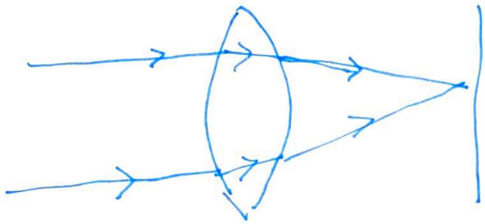
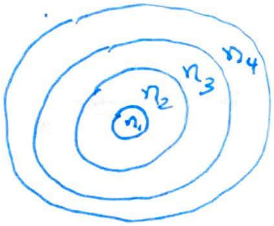
Geometrical Optics

Light before Maxwell — Rays.



Also an actual optical fiber is graded



$n_1 > n_2 > n_3 \cdots$

$n(r)$

We assume $n(r)$, $\epsilon(r)$ vary on a scale $l \gg \lambda$, wavelength of light.

$\boxed{\text{Analog of WKB in quantum mechanics}}$

$$\vec{E}(\vec{r},t) = \vec{E}_0\, e^{i\omega S(r)/c}\, e^{-i\omega t}$$

$$\vec{B}(\vec{r},t) = \vec{B}_0\, e^{i\omega S(r)/c}\, e^{-i\omega t}$$

$S(\vec{r})$ is the "eikonal"

$\Longleftrightarrow$ WKB in QM.

Then Maxwell's equations become

$$\boxed{\vec{E}_0 \cdot \vec{\nabla} S = 0} \qquad (\nabla \cdot D = 0)$$

We ignore $\vec{\nabla}\epsilon$ because $d \gg \lambda$

$$\boxed{\vec{B}_0 \cdot \vec{\nabla} S = 0} \qquad (\nabla \cdot B = 0)$$

$$\left(\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}\right)$$

$$\frac{1}{c} \vec{\nabla} S \times \vec{E}_0 = \vec{B}_0$$

$$\frac{1}{c} \vec{\nabla} S \times \vec{B}_0 = -\mu\epsilon \vec{E}_0 \qquad \left(\vec{\nabla} \times \vec{B} = \mu\epsilon \frac{\partial \vec{E}}{\partial t}\right)$$

Consistency of last 2 equations leads to

$$\vec{\nabla} S \cdot \vec{\nabla} S = c^2 \mu \epsilon(r)$$

or

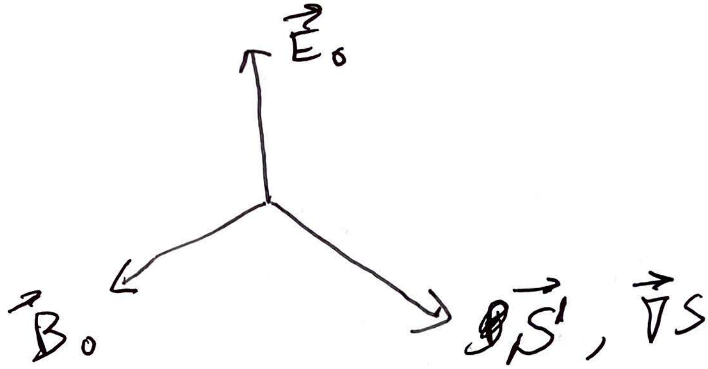
$$\boxed{\vec{\nabla} S \cdot \vec{\nabla} S = n^2(r)}$$

Eikonal equation.

$n(r) \rightarrow$ refractive index.

Poynting vector

$$\vec{S} = \frac{1}{\mu} \vec{E} \times \vec{B} = \frac{1}{2\mu c} |E_0|^2 \vec{\nabla} S$$

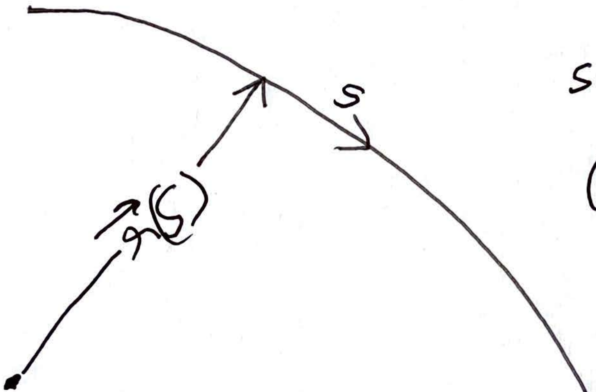


We write

$$\vec{\nabla} S = n(\vec{r})\, \hat{k}(\vec{r})$$

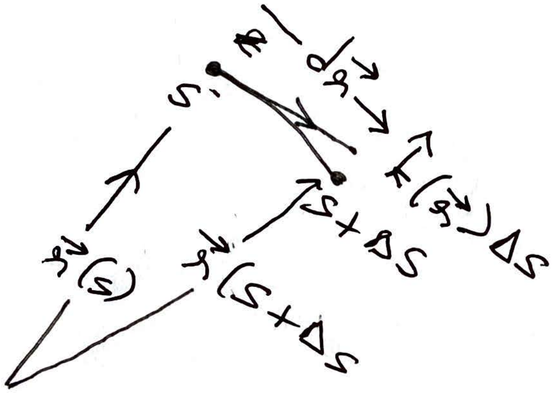
$\hat{k}(\vec{r}) \longrightarrow$ unit vector in the direction of the ray

Path of ray
$\vec{r}(s)$
$s \rightarrow$ path length along ray
(analogous to proper time in relativity)



So $d\vec{r}^{\,2} = ds^2$

i.e. $\left| \dfrac{d\vec{r}}{ds} \right| = 1$.



We have $d\vec{r} = \hat{k}(\vec{r})\,\Delta s$

Or $\dfrac{d\vec{r}}{ds} = \hat{k}(\vec{r})$ (OK because $\left|\dfrac{d\vec{r}}{ds}\right| = |\hat{k}(\vec{r})| = 1$).

So $\vec{\nabla} S = n(\vec{r})\dfrac{d\vec{r}}{ds}$

Therefore $\dfrac{d}{ds}\left(n(\vec{r})\dfrac{d\vec{r}}{ds}\right) = \vec{\nabla}\dfrac{dS}{ds}$

But $\dfrac{d}{ds} = \hat{k}\cdot\vec{\nabla}$ so $\dfrac{dS}{ds} = \hat{k}\cdot\vec{\nabla}S$

$= \hat{k}\cdot\hat{k}\,n(r)$

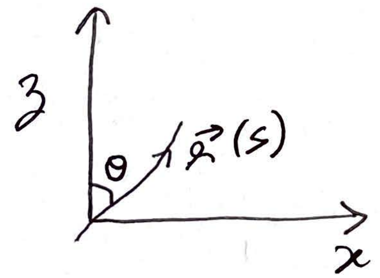
So finally we obtain the ray equation

$$\boxed{\frac{d}{ds}\left[n(\vec{r})\,\frac{d\vec{r}}{ds}\right] = \vec{\nabla} n(r)}$$

(i) Allows solutions with $\left|\frac{d\vec{r}}{ds}\right| = 1$ for all s.

(ii) Generalization of Snell's law.

Suppose $\vec{\nabla} n(\vec{r})$ is a vector in the $\hat{z}$ direction.



Then $n(\vec{r})\,\frac{dx}{ds} = \text{Constant}$

or $\boxed{n \sin\theta = \text{Constant}}$

Why is $\left|\frac{d\vec{r}}{ds}\right| = 1$ for ray equation?

$$\frac{d}{ds}\left(\frac{d\vec{r}}{ds}\right)^2 = 0$$

$$\Rightarrow \quad \frac{d^2\vec{r}}{ds}\cdot\frac{d\vec{r}}{ds} = 0.$$

The ray equation preserves this condition

LHS =
$$\frac{d}{ds}\left[n(\vec{r}(s))\frac{d\vec{r}}{ds}\right]$$
$$= \left(\vec{\nabla}n\cdot\frac{d\vec{r}}{ds}\right)\frac{d\vec{r}}{ds} + n(\vec{r})\frac{d^2\vec{r}}{ds^2}$$

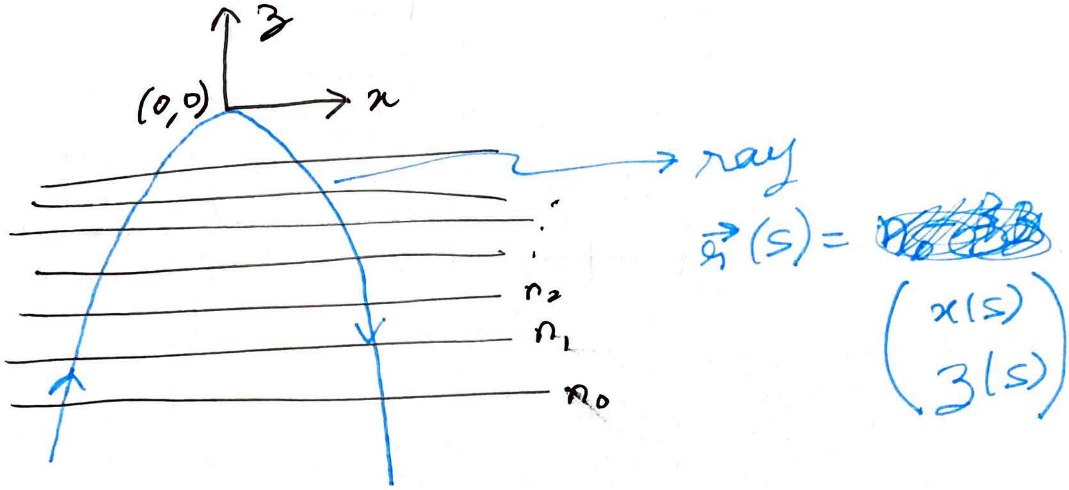
RHS = $\vec{\nabla}n(\vec{r})$

$$(LHS - RHS)\cdot\frac{d\vec{r}}{ds} = \left(\vec{\nabla}n\cdot\frac{d\vec{r}}{ds}\right)\frac{d\vec{r}}{ds}\cdot\frac{d\vec{r}}{ds} + n(\vec{r})\frac{d^2\vec{r}}{ds^2}\cdot\frac{d\vec{r}}{ds}$$
$$- \frac{d\vec{r}}{ds}\cdot\vec{\nabla}n$$
$$= n(\vec{r})\frac{d^2\vec{r}}{ds^2}\cdot\frac{d\vec{r}}{ds} = 0$$

QED

Reflection by a graded dielectric

$$n(\vec{r}) = n_0 - \beta z$$



Ray Equation

$$\frac{d}{ds}\left[\vec{n}(\vec{r}(s))\,\frac{d\vec{r}}{ds}\right] = -\beta\begin{pmatrix}0\\1\end{pmatrix} \qquad (\star)$$

Boundary condition $\vec{r}(0) = \begin{pmatrix}0\\0\end{pmatrix}$

and $\dfrac{d\vec{r}}{ds}(0) = \begin{pmatrix}0\\1\end{pmatrix}$.

Integrating $(\star)$ and imposing boundary conditions

$$\frac{dx}{ds} = \frac{n_0}{n_0 - \beta z} \; ; \quad \frac{dz}{ds} = \frac{-\beta s}{n_0 - \beta z}$$

Solution for $z(s)$

$$\boxed{z(s) = \frac{1}{\beta}\left[n_0 - \sqrt{n_0^2 + \beta^2 s^2}\right]}$$

Plug into equation for $x(s)$ and solve

$$\boxed{x(s) = \frac{n_0}{\beta} \ln\left(\frac{\beta s + \sqrt{n_0^2 + \beta^2 s^2}}{n_0}\right)}$$

Verify that (magically?)

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1.$$