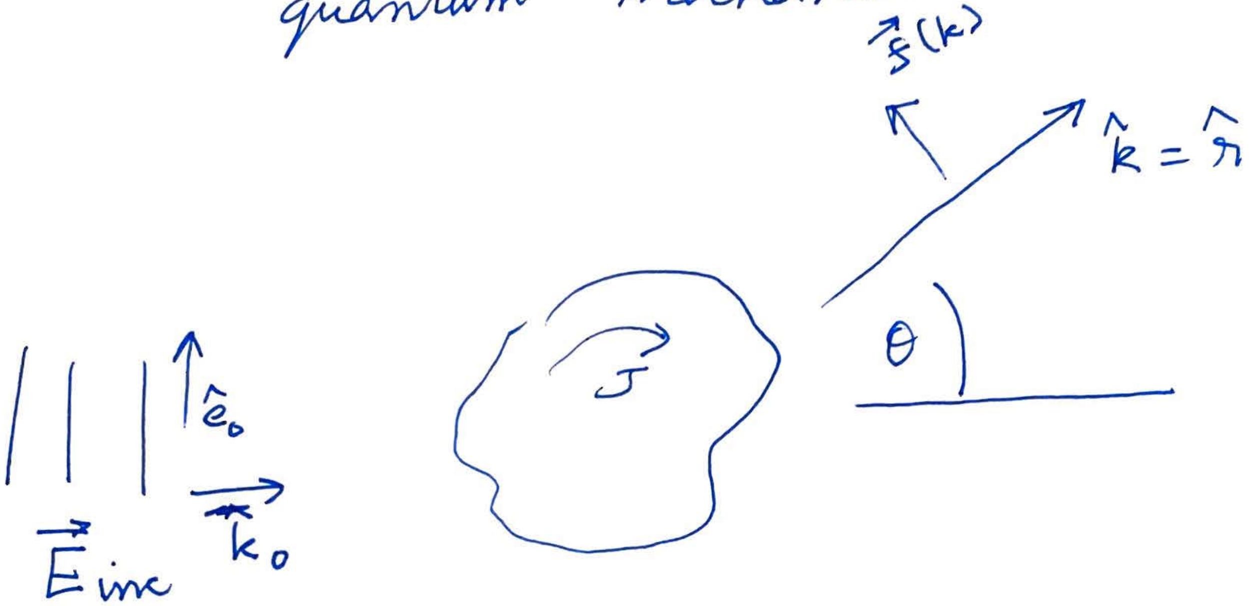
Scattering

Radius of curvature of matter $\lesssim$ Wavelength of radiation.

Similar to scattering problems in quantum mechanics



At large r

$$\vec{E}(\vec{r},t) = E_0 \left[\hat{e}_0 e^{i\vec{k}_0 \cdot \vec{r}} + \frac{e^{ikr}}{r} \vec{f}(\vec{k}) \right] e^{-i\omega t}$$

Note $\vec{f}(\vec{k})$ is a complex vector function of $\vec{k}$

Differential scattering cross-section

$$\frac{d\sigma}{d\Omega} = \frac{\text{scattered power per unit solid angle}}{\text{incident power per unit area}}$$

or $$\frac{d\sigma}{d\Omega} = \frac{r^2\,\hat{r}\cdot\langle\vec{S}_{rad}\rangle}{|\langle\vec{S}_{inc}\rangle|}$$

$\vec{S} \rightarrow$ Poynting vector.

For a polarization $\vec{\epsilon}$

$$\left.\frac{d\sigma}{d\Omega}\right|_{\vec{\epsilon}} = r^2\,\frac{|\epsilon^*\cdot\vec{E}_{rad}|^2}{|E_0|^2} = |\epsilon^*\cdot f(k)|^2.$$

$\vec{E}_{rad}$ is produced by some induced current $\vec{J}$.

Recall expression for radiation from a current $\vec{J}$

$$\vec{A}(\vec{r},t) = \frac{\mu_0 e^{-i\omega t}}{4\pi r}\int d^3r'\,\vec{J}(r') \times e^{i(kr - \vec{k}\cdot\vec{r}')}$$

(Recall $\vec{k} = k\hat{r}$)

Using Maxwell's equations we obtain at large r

$$\vec{B}_{rad} = \frac{i\omega}{c}\,\hat{r}\times\vec{A} \quad (\text{from } \vec{B} = \vec{\nabla}\times\vec{A})$$

$$\vec{E}_{rad} = -i\omega\,(\hat{r}\times(\hat{r}\times\vec{A}))$$

(from ~~$\nabla\times E = -\frac{\partial B}{\partial t}$~~ $\nabla\times B = \frac{1}{c^2}\frac{\partial E}{\partial t}$)

So

$$\vec{E}(\vec{r},t) = -\frac{ik}{4\pi\epsilon_0 c}\,\hat{k}\times\left[\hat{k}\times\int d^3r'\,\vec{J}(\vec{r}')\,e^{-i\vec{k}\cdot\vec{r}'}\right] \times \frac{e^{i(kr-\omega t)}}{r}$$

So

$$\boxed{\frac{d\sigma}{d\Omega} = \left(\frac{k}{4\pi\epsilon_0 E_0 c}\right)^2 \left| \hat{k} \times \int d^3r' \, \vec{J}'(r) \, e^{-i\vec{k}\cdot\vec{r}'} \right|^2}$$

Thomson scattering

Single electron of charge e and mass m.

EOM $\quad m \frac{d^2\vec{r}_e}{dt^2} = -e\vec{E}(t)_{\text{incident}}$

$$\vec{E}_{\text{incident}} = \vec{e}_0 E_0 \, e^{i(\vec{k}_0\cdot\vec{r}_e - \omega t)}$$

$$\approx \vec{e}_0 E_0 \, e^{i\omega t}$$

Assuming $\vec{r} \simeq 0$ and $kr \ll 1$

i.e. electron does not move on the scale of a wavelength.

Under these conditions we can just use the formula for dipole radiation

$$\vec{E}_{rad} = -\frac{\mu_0}{4\pi}\omega^2 \left[\hat{k}\times(\hat{k}\times\vec{p})\right] \frac{e^{i(kr-\omega t)}}{r}$$

where the dipole moment $\vec{p} = -e\vec{r}_e$

obeys

$$-m\omega^2\vec{p} = e^2\vec{e}_0 E_0$$

So cross-section for all scattered polarizations

$$\left(\frac{d\sigma}{d\Omega}\right) = \left(\frac{e^2}{4\pi \epsilon_0 m c^2}\right)^2 |\hat{k} \times \vec{e}_0|^2$$

$$\equiv r_e^2 \left(1 - |\hat{k} \cdot e_0|^2\right)$$

$$r_e \equiv \frac{e^2}{4\pi \epsilon_0 m c^2} = 2.82 \times 10^{-15}\ \mathrm{m}$$

is the classical electron radius.

The cross-section for scattering radiation with incoming polarization $\vec{e}_0$ into outgoing polarization $\vec{e}_s$ is

$$\frac{d\sigma}{d\Omega} = r_e^2 \left| \vec{e}_s \cdot (\hat{k} \times (\hat{k} \times \vec{e}_0)) \right|^2$$

$$= \underline{\underline{r_e^2 \, |\vec{e}_s \cdot \vec{e}_0|^2}} \quad \left(\text{using vector identities and } \hat{k} \cdot \vec{e}_s = 0\right).$$

(Note: there are errors in Zangwill on these results!)

Rayleigh Scattering

EM scattering off ~~dst~~ dielectric spheres

$$\vec{p} = \alpha \epsilon_0 \vec{E}$$

$\alpha \rightarrow$ polarizability

So $$\frac{d\sigma}{d\Omega} = \left(\frac{k_0^2 \alpha}{4\pi}\right)^2 |\vec{e}_s \cdot \vec{e}_0|^2$$

$$\alpha = \# a^3$$ where is the sphere radius

So $$\frac{d\sigma}{d\Omega} \sim (k_0 a)^4 a^2$$

$\frac{1}{\lambda^4}$ dependence on wavelength

Why sky is blue!