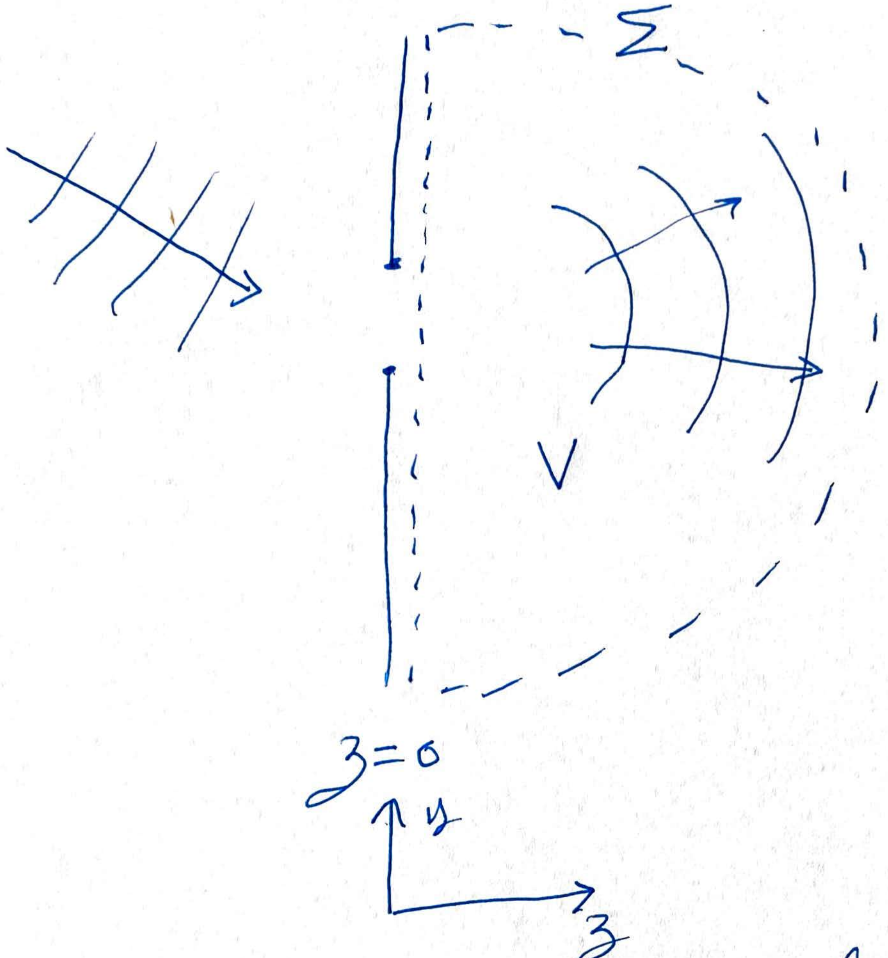
Diffraction



We consider scalar waves first

$$(\nabla^2 + k_0^2)\, u = 0 \quad ; \quad k_0 = \omega/c$$

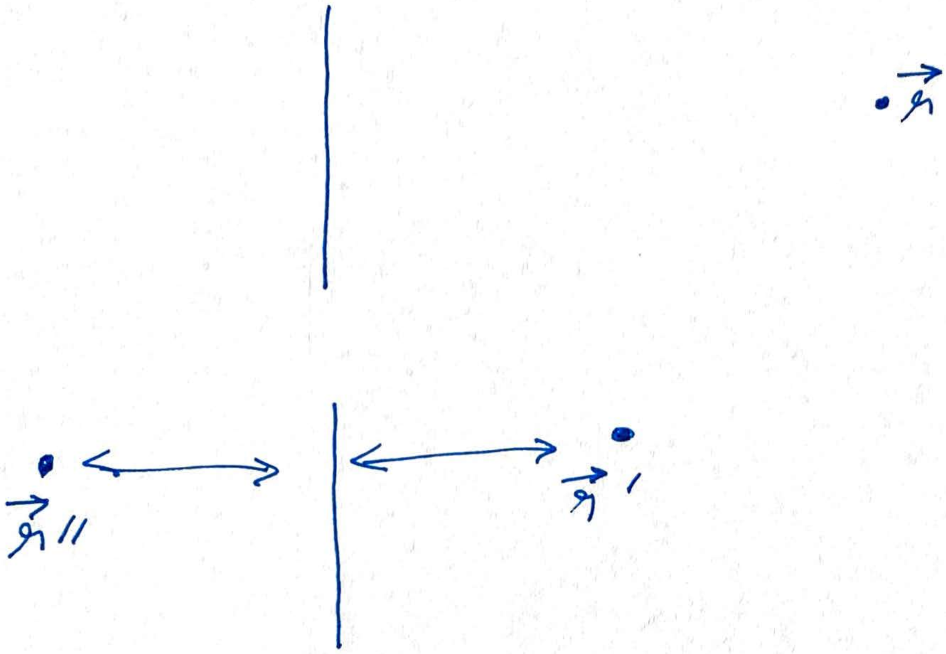
applies to sound, water waves, and each component of the electric and magnetic field.

We will use Green's functions to relate $u(\vec{r})$ at $z=0$ to $u(\vec{r} \to \infty)$.

Green's function

We solve $\left(\nabla^2 + k_0^2\right) G(\vec{r}, \vec{r}')$

$$= -\delta(\vec{r} - \vec{r}')$$



$z = 0$

subject to the boundary conditions

(i) $G(\vec{r}, \vec{r}') = 0$ when $z = 0$

(ii) $G(\vec{r}, \vec{r}') \longrightarrow \dfrac{e^{ik_0 r}}{4\pi r}$ as $|\vec{r}| = r \to \infty$

(outgoing waves).

Solution:– use method of images

$$G(\vec{r}, \vec{r}') = \frac{e^{ik_0|\vec{r} - \vec{r}'|}}{4\pi|\vec{r} - \vec{r}'|} - \frac{e^{ik_0|\vec{r} - \vec{r}''|}}{4\pi|\vec{r} - \vec{r}''|}$$

where $\vec{r}''$ is the image of $\vec{r}'$ for a mirror at $z = 0$.

Next we use Green's second identity

$$\int_V d^3r \left[\phi \nabla^2 \psi - \psi \nabla^2 \phi\right] = \int_{\Sigma} d\vec{S} \cdot \left[\phi \vec{\nabla} \psi - \psi \vec{\nabla} \phi\right]$$

Immediate consequence of Gauss's law.

We apply this identity to

$$\phi(\vec{r}) = u(\vec{r})$$

and $\psi(\vec{r}) = G(\vec{r}, \vec{r}\,')$.

Then LHS $= -u(\vec{r}\,')$

and RHS

$$= \int_{\Sigma} d\vec{S} \cdot \left[u(\vec{r}) \cdot \vec{\nabla} G(\vec{r}, \vec{r}\,') - G(\vec{r}, \vec{r}\,') \nabla u(\vec{r})\right]$$

We can verify later after determining $u(\vec{r})$ that the integral at $r \to \infty$ vanishes. So the integral over Σ reduces to an integral over the plane at $z = 0$.

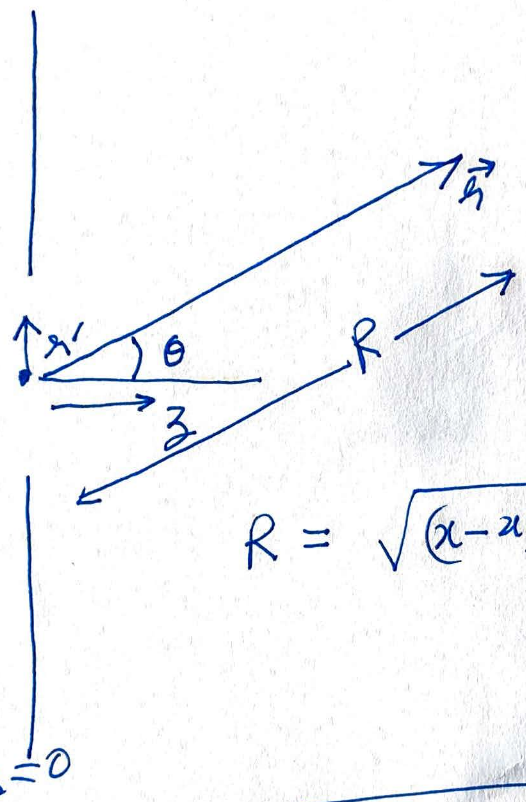
Also recall $G(\vec{r}, \vec{r}\,') = 0$ at $z = 0$

and $d\vec{S} \sim -\hat{z}$

and $$\frac{\partial}{\partial z}\left(\frac{e^{ik_0|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}\right)\Bigg|_{\text{at } z=0} = -\frac{\partial}{\partial z}\left(\frac{e^{ik_0|\vec{r}-\vec{r}\,''|}}{|\vec{r}-\vec{r}\,''|}\right).$$

Putting everything together we obtain the Rayleigh-Sommerfeld diffraction integral (after $\vec{r} \leftrightarrow \vec{r}\,'$)

$$u(\vec{r}) = -\frac{1}{2\pi} \int_{z'=0} dS'\, u(\vec{r}\,') \frac{\partial}{\partial z} \left[\frac{e^{ik_0|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} \right]$$



$$R = \sqrt{(x-x')^2 + (y-y')^2 + z^2}$$

$$\boxed{u(\vec{r}) = -\frac{1}{2\pi} \int d^2r'\, u(\vec{r}\,') \left(ik_0 - \frac{1}{R} \right) \frac{e^{ik_0R}}{R} \frac{z}{R}}$$

Fraunhofer diffraction

So far everything is exact.

Now assume

(i) $u(\vec{r}) = 0$ at $z=0$ except in an aperture of size d

(ii) Far field $r \gg d, \lambda$.

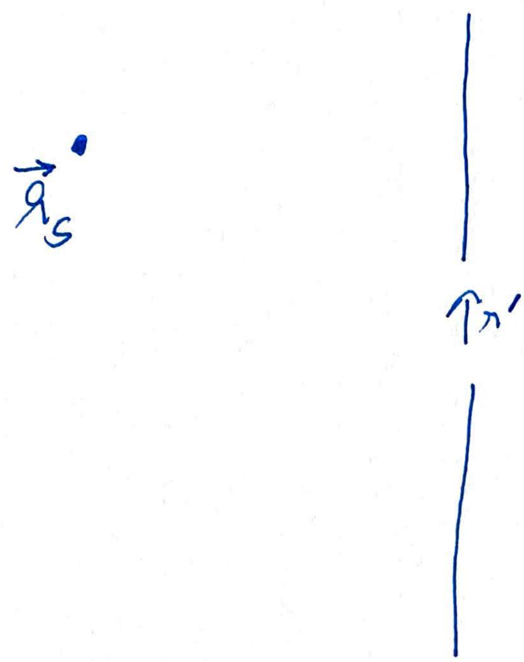
Then

$$R = |\vec{r} - \vec{r}'| \approx r - \hat{r}\cdot\vec{r}'$$

and define $k_0 \hat{r} = \vec{k}$

$$u(\vec{r}) = -\frac{ik_0}{2\pi}\frac{e^{ik_0 r}}{r}\cos\theta \underbrace{\int_{\text{aperture}} d^2r'\, u(\vec{r}')\, e^{-i\vec{k}\cdot\vec{r}'}}_{\text{Fourier transform of aperture.}}$$

What is $u(\vec{r}')$ in the aperture?



We assume there is a point source at $\vec{r}_s$.

Then $$u(\vec{r}') = A \frac{e^{ik_0|\vec{r}'-\vec{r}_s|}}{4\pi|\vec{r}'-\vec{r}_s|}$$

(incoming wave)

Assuming $r_s \gg \lambda, d$ we again use

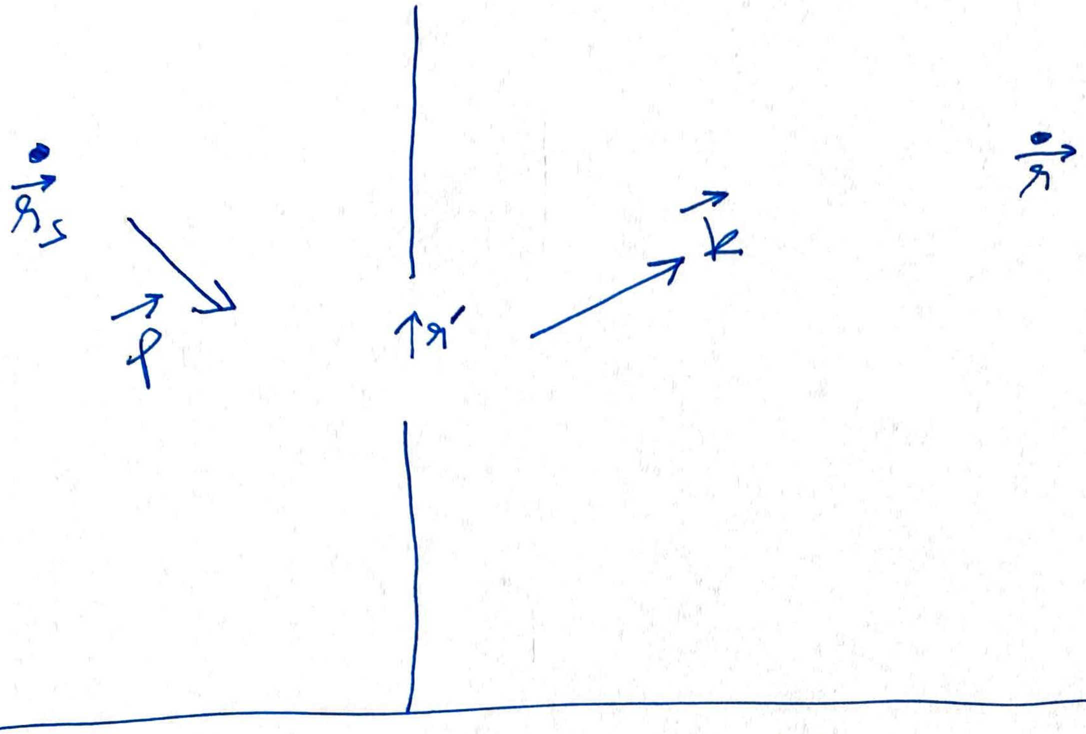
$$|\vec{r}_s - \vec{r}'| \approx r_s - \hat{r}_s \cdot \vec{r}'$$

and write $k_0 \hat{r}_s = -\vec{p}$

incoming wavevector.

Then $$u(\vec{r}') = \frac{A e^{ik_0 r_s}}{4\pi r_s} e^{i\vec{p}\cdot\vec{r}'}$$

$$|\vec{p}| = |\vec{k}| = k_0 = \omega/c$$

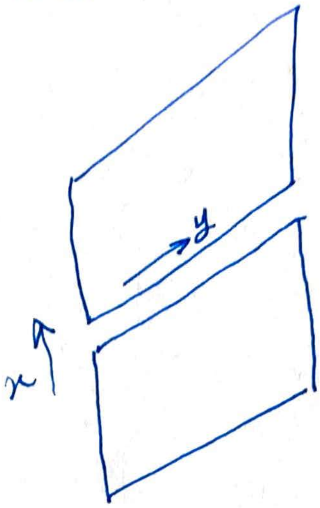


$$\boxed{u(\vec{r}) = \left[\frac{A\cos\theta\, e^{ik_0(r_s+r)}}{8\pi^2 r\, r_s}(-ik_0)\right] \times \int_{\text{aperture}} d^2 r_\perp\, e^{i(\vec{p}-\vec{k})\cdot r_\perp}}$$

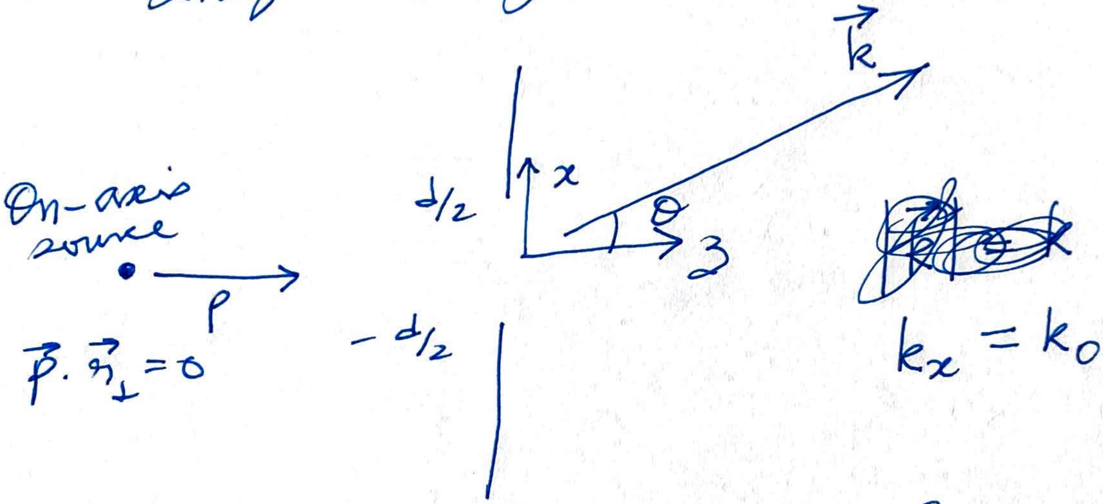
Fraunhofer diffraction

$$\text{Intensity} \propto \left| \int_{\text{aperture}} d^2 r_\perp\, e^{i(\vec{p}-\vec{k})\cdot r_\perp} \right|^2$$

(I) Planar slit



Integral over y implies $p_y = k_y$



$\vec{p}\cdot\vec{r}_\perp = 0$

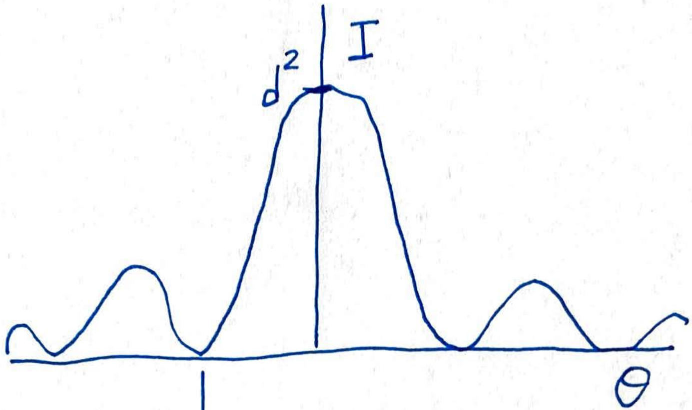
$k_x = k_0 \sin\theta$

$$I \propto \left| \int_{-d/2}^{d/2} dx \, e^{-i k_0 \sin\theta \, x} \right|^2$$

$$= \frac{4 \sin^2 (k_0 \sin\theta \, d/2)}{k_0^2 \sin^2\theta}$$

For small θ, $\sin\theta \sim \theta$

$$I \propto \frac{4\sin^2(k_0 d\theta/2)}{k_0^2\theta^2}$$



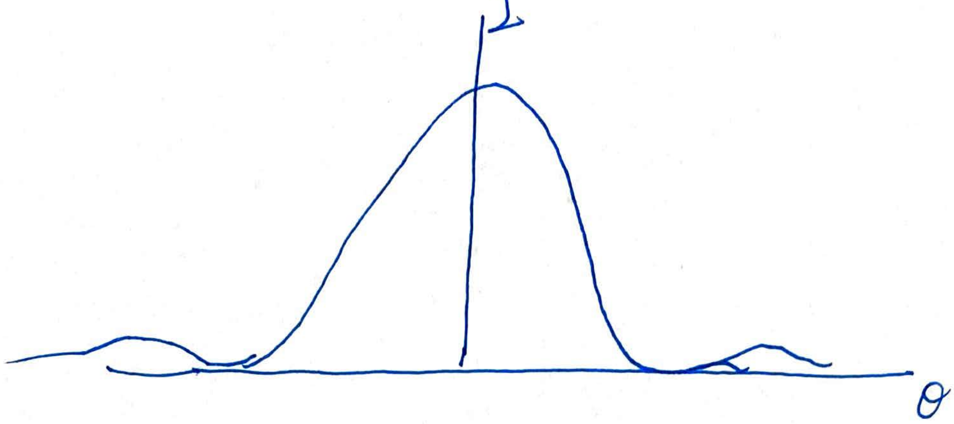
$$\theta_0 = \frac{2\pi}{k_0 d} = \frac{\lambda}{d} \rightarrow \text{diffraction angle}$$

(II) Circular slit of radius a

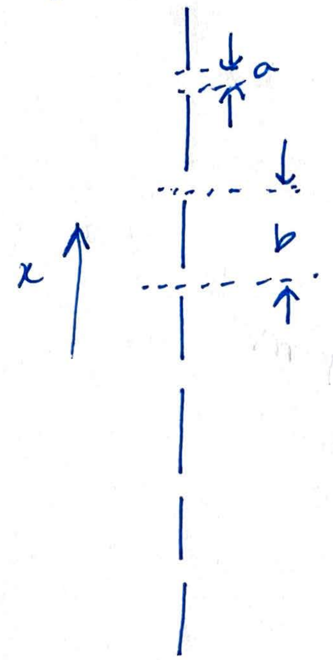
$$I \propto \left| \int_0^a d^2 r_\perp \, e^{i\vec{k}\cdot r_\perp} \right|^2$$

$$= \left| \int_0^a r\,dr \int_0^{2\pi} d\phi \, e^{i k_0 \sin\theta \, r \cos\phi} \right|^2$$

$$= \frac{4\pi^2 a^2 J_1^2(k_0^2 a^2 \sin^2\theta)}{k_0^2 \sin^2\theta}$$



(III) Diffraction grating



1+N slits of width a with spacing b

Apertures

$x = nb + s$

$-a/2 < s < a/2$

$n = 0, 1, \ldots N$

Then

$$I \propto \left| F(k_0 \sin\theta) \right|^2$$

where

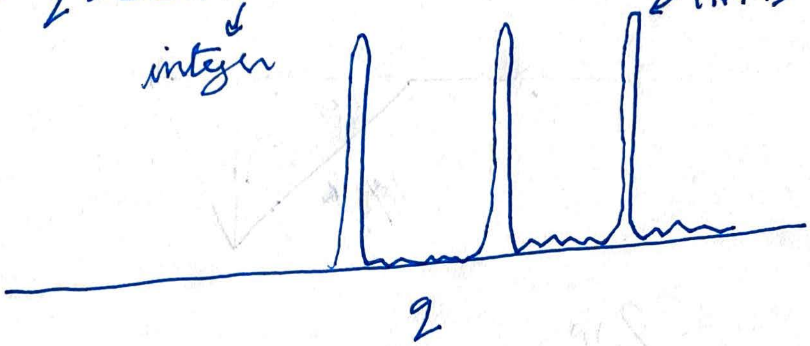
$$F(q) = \sum_{n=0}^{N} e^{-iqnb} \int_{-a/2}^{a/2} ds\, e^{-iqs}$$

$$= \left(\frac{1 - e^{-iqb(N+1)}}{1 - e^{iqb}} \right) \left(\frac{e^{-iqa/2} - e^{iqa/2}}{-iq} \right).$$

Then

$$|F(q)|^2 = \left[\frac{4\sin^2(qa/2)}{q^2}\right] \times \left[\frac{\sin^2\left[q\frac{b(N+1)}{2}\right]}{\sin^2\left(\frac{qb}{2}\right)}\right].$$

Peaks $\propto N^2$
at $qb = 2\pi m$
integer

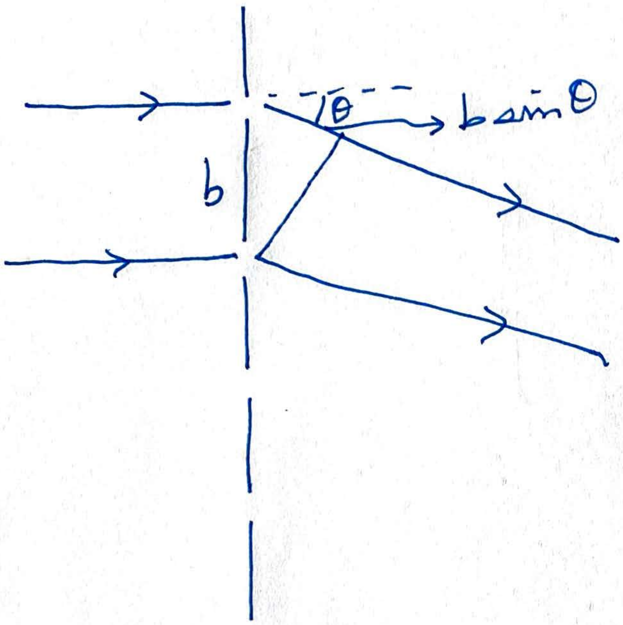


Envelope varies
on scale $q \sim 1/a$
Peaks have width
$q \sim 1/b$.

Peaks of strength N^2 when the Bragg condition is obeyed

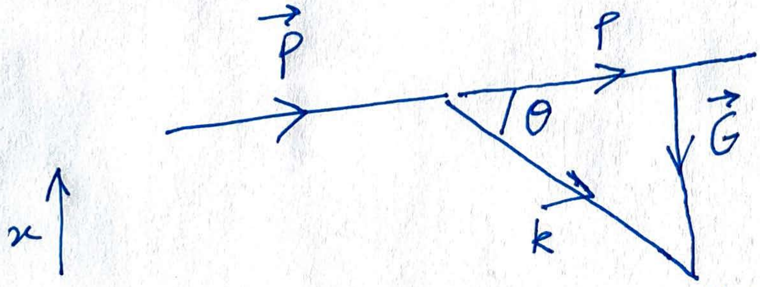
$$k_0 \sin\theta \, b = 2\pi m$$

or $$\boxed{b \sin\theta = m\lambda}$$



Constructive interference of outgoing waves when $b\sin\theta = m\lambda$.

Alternative interpretation



Wavevector transfer by grating in x direction = G

$$G = |k| \sin\theta = \frac{2\pi}{\lambda} \sin\theta = \frac{2\pi m}{b}$$

reciprocal lattice vector!

Vector diffraction

Although the components of $\vec{E}$ and $\vec{B}$ all satisfy the Helmholtz equation, the scalar diffraction formula does not satisfy Maxwells equations (which are more restictive). In general, we have to solve Maxwells equations and use boundary conditions on all the fields. The resulting diffraction formula (see Zangwill) replacing the Rayleigh-Sommerfeld formula is

$$\vec{E}(\vec{r}) = 2\vec{\nabla}\times \int_{z'=0} d^2\vec{r}_\perp' \left[\hat{z}\times \vec{E}(\vec{r}_\perp')\right] \frac{e^{ikR}}{4\pi R}$$

and

$$\vec{B}(\vec{r}) = 2\vec{\nabla}\times \int_{z'=0} d^2 r_\perp' \left[\hat{z}\times \vec{B}(\vec{r}_\perp')\right] \frac{e^{ikR}}{R}$$