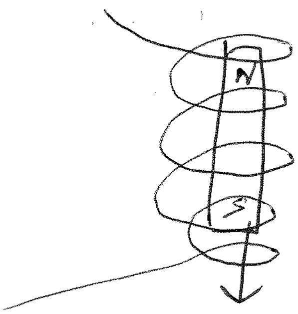
Time dependence:

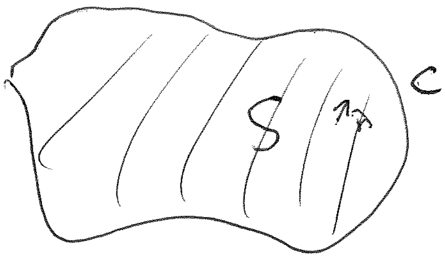
Maxwell's Equations

Faraday's law of induction



Moving magnets in coils induces current in the wire.

$\Rightarrow$ Electromotive force $= -$ rate of change of flux



$$\oint \vec{E} \cdot dl = - k_4 \frac{d}{dt} \int_S \vec{B} \cdot \hat{n} \, da$$

Lenz's law $\rightarrow$ the current creates a magnetic field which opposes the change in flux.

$$\int_S (\vec{\nabla} \times \vec{E}) \cdot \hat{n} \, da$$

$$\Rightarrow \quad \boxed{\vec{\nabla} \times \vec{E} = - k_4 \frac{\partial \vec{B}}{\partial t}}$$

Maxwell:

Ampere's law cannot be true in general because it violates the conservation of current.

$$\vec{\nabla} \times \vec{B} = 4\pi k_3 \vec{J}$$

$$\Rightarrow \quad \vec{\nabla} \cdot \vec{J} = 0 \neq -\frac{\partial \rho}{\partial t}.$$

Maxwell's correction

$$\vec{\nabla} \times \vec{B} = 4\pi k_3 \vec{J} + \vec{T}$$

$$\Rightarrow \quad \vec{\nabla} \cdot \vec{T} = 4\pi k_3 \frac{\partial \rho}{\partial t}$$

but $\quad 4\pi k_1 \rho = \vec{\nabla} \cdot \vec{E}$

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} = \frac{1}{4\pi k_1} \vec{\nabla} \cdot \frac{\partial \vec{E}}{\partial t}$$

So choose

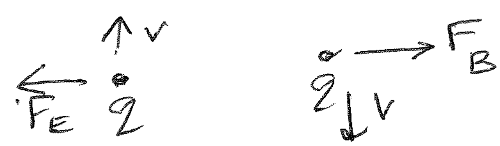
$$\vec{T} = \frac{k_3}{k_1} \frac{\partial \vec{E}}{\partial t}.$$

Maxwell's Equations

$$\boxed{\begin{aligned} \nabla \times B &= 4\pi k_3 \vec{J} + \frac{k_3}{k_1}\frac{\partial \vec{E}}{\partial t} \\ \vec{\nabla}\times\vec{E} &= -k_4 \frac{\partial B}{\partial t} \\ \nabla \cdot B &= 0 \\ \nabla \cdot E &= 4\pi k_1 \rho \end{aligned}}$$

Lorentz force law on a charge q with velocity $\vec{V}$

$$\boxed{\vec{F} = q\vec{E} + \frac{k_2}{k_3} q \vec{V}\times\vec{B}}$$



$$\frac{F_B}{F_E} = \frac{k_2}{k_1} V^2$$

and indeed $\frac{k_1}{k_2} = c^2$. (by Lorentz invariance)

Also wave eqn gives us $\frac{1}{c^2}$. $(\vec{J} = \rho = 0)$

$$\nabla \times B = \frac{k_3}{k_1}\frac{\partial E}{\partial t} \quad (1), \qquad \nabla\times E = -k_4 \frac{\partial \vec{B}}{\partial t} \quad (2)$$

Using $\nabla\times$ of (1)

$$\Rightarrow \left(\nabla^2 - \frac{k_4 k_3}{k_1}\frac{\partial^2}{\partial t^2}\right)\vec{B} = 0.$$

$$\boxed{\frac{1}{c^2} = \frac{k_4 k_3}{k_1}}$$

Constants	k_1	k_2	k_3	k_4
CGS	1	$1/c^2$	$1/c$	$1/c$
MKS	$\dfrac{1}{4\pi\epsilon_0}$	$\dfrac{\mu_0}{4\pi}$	$\dfrac{\mu_0}{4\pi}$	1

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}.$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\vec{F} = Q(\vec{E} + \vec{v} \times \vec{B})$$

$$\mu_0 \equiv 4\pi \times 10^{-7} \frac{N}{A^2}$$

(fixes definition of ampere).

Vector Potentials

Can reduce Maxwell's equations ~~for~~ to 2 eqns by introducing vector potentials.

$$\vec{\nabla}\cdot\vec{B} = 0 \implies \boxed{\vec{B} = \vec{\nabla}\times\vec{A}}$$

$\vec{A}$ is the vector potential.

Now $\vec{\nabla}\times\vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$

So we no longer have $\vec{E} = -\nabla\Phi$.

Rather

$$\boxed{\vec{E} = -\nabla\vec{\Phi} - \frac{\partial \vec{A}}{\partial t}}$$

works!

Remaining Maxwell's equations are

$$\nabla\times B = \mu_0 J + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

$$\Rightarrow \boxed{-\nabla^2\vec{A} + \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) = \mu_0 J - \frac{1}{c^2}\vec{\nabla}\frac{\partial \Phi}{\partial t} - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2}}$$

and

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$$

$$\Rightarrow \boxed{-\nabla^2\Phi - \nabla\cdot\frac{\partial A}{\partial t} = \rho/\epsilon_0}$$

Gauge transformation

$$\boxed{\begin{aligned} \vec{A} &= \vec{A} + \vec{\nabla}\Lambda(x,t) \\ \Phi &= \Phi - \frac{\partial \Lambda}{\partial t} \\ &\text{leaves } \vec{E} \text{ and } \vec{B} \text{ unchanged} \end{aligned}}$$

Choose $\Lambda(x,t)$ to simplify the 2 Maxwell equations.

Lorentz Gauge

Choose $\nabla \cdot A + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0.$

Suppose this condition is not satisfied
Then under a gauge transformation

$$\vec{A}' = A + \vec{\nabla}\Lambda$$

$$\Phi' = \Phi - \frac{\partial \Lambda}{\partial t}$$

implies

$$\nabla \cdot A' + \frac{1}{c^2}\frac{\partial \Phi'}{\partial t} = \nabla \cdot A + \nabla^2 \Lambda + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} - \frac{1}{c^2}\frac{\partial^2 \Lambda}{\partial t^2} = 0$$

or $$\nabla^2 \Lambda - \frac{1}{c^2}\frac{\partial^2 \Lambda}{\partial t^2} = -\nabla \cdot \Lambda - \frac{1}{c^2}\frac{\partial \Phi}{\partial t}$$

This equation can always be solved for Λ.

In the Lorentz gauge the Maxwell's equations become

$$-\nabla^2\Phi - \nabla\cdot\frac{\partial\vec{A}}{\partial t} =$$

$$= \boxed{-\nabla^2\Phi + \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = \frac{\rho}{\epsilon_0}}$$

and

$$-\nabla^2\vec{A} + \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = \mu_0\vec{J} - \vec{\nabla}\underbrace{\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right)}_{=0}$$

$$\boxed{-\nabla^2\vec{A} + \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = \mu_0\vec{J}}$$

$$\Box = -\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

D'Alembertian operator.

$$\Box\Phi = \rho/\epsilon_0\,, \qquad \Box\vec{A} = \mu_0\vec{J}$$

~~Col~~ Coulomb Gauge

$$\vec{\nabla}\cdot\vec{A} = 0.$$

Can always be satisfied

Choose $\vec{A}' = \vec{A} + \vec{\nabla}\Lambda$

Then

$$\vec{\nabla}\cdot\vec{A}' = \nabla\cdot\vec{A} + \nabla^2\Lambda = 0.$$

This equation can be solved for Λ.

Maxwell ~~eql~~ equation (I) becomes

$$\text{(A)}\quad \boxed{-\nabla^2\Phi = \rho/\epsilon_0}$$

exactly as in electrostatics.

The second Maxwell's equation becomes

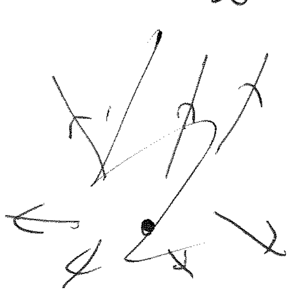
$$\text{(B)}\quad \boxed{\Box\vec{A} = \mu_0\vec{J} - \frac{1}{c^2}\vec{\nabla}\frac{\partial\Phi}{\partial t}}$$

(A) has the same solution as in electrostatics

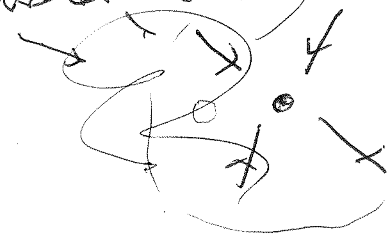
$$\Phi(x,t) = \int d^3\vec{x}'\,\frac{\rho(x',t)}{4\pi\epsilon_0\,|\vec{x}-x'|}$$

Verify by using $\nabla^2\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = -4\pi\delta(\vec{x}-\vec{x}')$.

Seems to imply instantaneous "action at a distance".



~~More story~~



Now the second Maxwell equation can be written

(B) $$\Box \vec{A} = \mu_0 \vec{J} - \frac{1}{4\pi\epsilon_0 c^2} \int d^3x' \, \frac{\partial \rho(x',t)}{\partial t} \times \vec{\nabla}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)$$

Use $\frac{\partial \rho}{\partial t} = -\nabla'_{\cdot} \vec{J}(x',t)$

and $\epsilon_0 c^2 = 1/\mu_0$

$$\Box \vec{A} = \mu_0 \vec{J} + \frac{\mu_0}{4\pi} \vec{\nabla}\left[\int d^3x' \, \frac{\nabla' \cdot \vec{J}(x',t)}{|\vec{x}-\vec{x}'|}\right]$$

RHS is a linear operator acting on $\vec{J}(\vec{x},t)$.

<u>Mathematical Aside</u>

Any vector field $\vec{S}(\vec{x})$ can be decomposed into longitudinal S_ℓ and transverse components S_t

$\vec{S} = \vec{S}_\ell + \vec{S}_t$ so that

$\vec{\nabla} \times \vec{S}_\ell = 0$ (like an electric field)

$\vec{\nabla} \cdot \vec{S}_t = 0$ (like a magnetic field)

Projection operators

$$\vec{S}_\ell(\vec{x}) = -\frac{1}{4\pi}\vec{\nabla}\int d^3x' \, \frac{\nabla'\cdot\vec{S}(x')}{|\vec{x}-\vec{x}'|}$$

and
$$\vec{S}_t(\vec{x}) = \frac{1}{4\pi}\vec{\nabla}\times\int d^3x' \, \frac{\vec{\nabla}'\times\vec{S}(\vec{x}')}{|\vec{x}-\vec{x}'|}.$$

Clearly these fields obey $\vec{\nabla}\times\vec{S}_\ell = 0$ and $\vec{\nabla}\cdot\vec{S}_t = 0$. Only need to prove

$$\vec{S}_\ell + \vec{S}_t = \vec{S}.$$

Equivalently we need to prove

$$\vec{\nabla}\cdot\vec{S}_\ell = \vec{\nabla}\cdot\vec{S} \quad \text{and} \quad \vec{\nabla}\times\vec{S}_t = \vec{\nabla}\times\vec{S}$$

$$\vec{\nabla}\cdot\vec{S}_\ell = -\frac{1}{4\pi}\nabla^2\int d^3x' \, \frac{\vec{\nabla}'\cdot\vec{S}}{|\vec{x}-\vec{x}'|} = \vec{\nabla}\cdot\vec{S}$$

The second relation leads to

$$\nabla\times\vec{S}_t = \frac{1}{4\pi}\nabla\times\nabla\times\int d^3x' \, \frac{\nabla'\times\vec{S}(x')}{|\vec{x}-\vec{x}'|}$$

$$= \underbrace{-\frac{\nabla^2}{4\pi}\int d^3x' \, \frac{\nabla'\times\vec{S}(x')}{|x-x'|}}_{\nabla\times\vec{S}}$$

$$+ \underbrace{\frac{1}{4\pi}\vec{\nabla}\int d^3x' \, (\nabla'\times S(x'))\,\vec{\nabla}\underbrace{\left(\frac{1}{|x-x'|}\right)}_{-\nabla'\left(\frac{1}{|x-x'|}\right)}}_{0 \text{ by "integration by parts"}}$$

Back to Maxwells eqn

$$\Box \vec{A} = \mu_0 \vec{J} + \frac{\mu_0}{4\pi} \vec{\nabla} \left[\int d^3_{x'} \frac{\nabla'\cdot J(x', t)}{|\vec{x} - \vec{x}'|} \right]$$

$$= \mu_0 (\vec{J}_l + \vec{J}_t) - \mu_0 \vec{J}_l$$

$$= \mu_0 \vec{J}_t .$$

Maxwell's Equations in Coulomb gauge.

$$\Box \vec{A} = \mu_0 \vec{J}_t \quad , \quad \nabla^2 \Phi = -\frac{\rho}{\epsilon_0} .$$