Review

Maxwell's Equations

$$\vec{\nabla}.\vec{E} = \rho/\epsilon_0$$ Coulomb's Law

$$\vec{\nabla}.\vec{B} = 0$$ No monopoles
Flux conservation

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$ Faraday's Law of Induction

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

Ampere's Law + Maxwell's displacement current.

Equations require

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}.\vec{J} = 0$$ Conservation of charge.

$$\mu_0 \equiv 4\pi \times 10^{-7}\ \frac{N}{A^2}$$ fixes definition of ampere.

$$c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$$ → velocity of light.

Lorentz force law

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}).$$

Gauge potentials

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}.$$

Exactly solves ~~sou~~ source-free equations.

Gauge invariance

$$\vec{A} \longrightarrow \vec{A} + \vec{\nabla}\Lambda$$

$$\Phi = \Phi - \frac{\partial \Lambda}{\partial t}.$$

Wave function

$$\psi \to \psi e^{iq\Lambda/\hbar}$$

Leave $\vec{E}$ and $\vec{B}$ invariant.

Lorentz Gauge

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0$$

Maxwell's Equations become

$$-\nabla^2 \Phi + \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = \rho/\epsilon_0$$

$$-\nabla^2 \vec{A} + \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = \mu_0 \vec{J}$$

Coulomb - Gauge

$$\vec{\nabla}\cdot\vec{A} = 0$$

Maxwell's equations become

$$-\nabla^2 \Phi = \rho/\epsilon_0$$

$$-\nabla^2 \vec{A} + \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = \mu_0 \vec{J}_t$$

$\vec{J}_t \rightarrow$ transverse component of $\vec{J}$

$$\vec{J} = \vec{J}_\ell + \vec{J}_t$$

Decomposition is simplest in Fourier space

$$\vec{J}(\vec{k}) = \int d^3 r\, \vec{J}(\vec{r})\, e^{-i\vec{k}\cdot\vec{r}}$$

$$\vec{J}(\vec{k}) = \vec{J}_\ell(\vec{k}) + \vec{J}_t(\vec{k})$$

$$J_{\ell i}(\vec{k}) = \frac{k_i k_j}{k^2} J_j(\vec{k})$$

$$J_{ti} = \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) J_{tj}$$

Verify that

$$k_i J_{ti} = 0 \Rightarrow \vec{\nabla} \cdot \vec{J}_t = 0$$

$$\epsilon_{ijm} k_j J_{lm} = 0 \Rightarrow \vec{\nabla} \times \vec{J}_l = 0.$$

Energy current

Poynting vector

$$\vec{S}_p = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

Momentum density

$$\vec{g} = \frac{\vec{S}_p}{c^2}$$

Electrostatics

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r' \, \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

Exact.

$$\approx \frac{Q}{4\pi\epsilon_0 r} + \frac{\vec{p}\cdot\vec{r}}{4\pi\epsilon_0 r^3} + \cdots$$

$$\vec{p} = \int d^3r \, \vec{r} \, \rho(\vec{r})$$: dipole moment.

$$Q = \int d^3r \, \rho(\vec{r})$$

Magnetostatics

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$\approx \frac{\mu_0}{4\pi} \frac{\vec{m}\times\vec{r}}{r^3}$$

$$\vec{m} = \frac{1}{2}\int d^3r \, \vec{r}\times\vec{J}(\vec{r})$$

: magnetic moment.

Time-dependent fields

Source $\boxed{\vec{J}(\vec{r},t) = \vec{J}(\vec{r})\, e^{-i\omega t}}$

Then

$$\left(-\nabla^2 - \frac{\omega^2}{c^2}\right)\vec{A} = \mu_0 \vec{J}$$

Inhomogeneous Helmholtz equation

Exact solution with outgoing waves in all directions

$$\boxed{\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} e^{-i\omega t} \int d^3r'\, \vec{J}(\vec{r}\,') \times \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}}$$

where $\boxed{k \equiv \omega/c}$.

3 length scales

$r \rightarrow$ observation point

$\lambda = 2\pi/k \rightarrow$ wavelength

$d =$ size of source.

Far field limit

$$r \gg d, \lambda.$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0 e^{i(kr-\omega t)}}{4\pi r} \times \left[\int d^3r' \, e^{-i\vec{k}\cdot\vec{r}'} \vec{J}(\vec{r}') \right]$$

where $\vec{k} = k\hat{r}$.

$$\vec{B} = \frac{i\omega}{c} \hat{r} \times \vec{A}$$

$$\vec{E} = -i\omega\left(\hat{r} \times (\hat{r} \times \vec{A})\right)$$

Dipole limit

$$\int d^3r' \, e^{-i\vec{k}\cdot\vec{r}'} \vec{J}(\vec{r}')$$

$$= -i\omega\vec{p} + ik(\hat{r} \times \vec{m}) - \omega \vec{k} \cdot \overleftrightarrow{Q}$$

quadrupole moment.

Relativity

$$x^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Raise and lower indices with

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}.$$

Lorentz ~~invar~~ invariant proper "distance"

$$ds^2 = dx^\mu dx_\mu$$

$$= c^2 dt^2 - dx^2 - dy^2 - dz^2$$

~~prod~~ proper time $d\tau = \frac{ds}{c}$

Four velocity $u^\mu = \frac{dx^\mu}{d\tau}$

$$u^\mu u_\mu = c^2.$$

Four momentum

$$p^\mu = m u^\mu = \begin{pmatrix} mc^2 \gamma \\ m\vec{v}\, \gamma \end{pmatrix}$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$J^\mu = \begin{pmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{pmatrix}$$

$$A^\mu = \begin{pmatrix} \Phi/c \\ A_x \\ A_y \\ A_z \end{pmatrix}$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$= \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

Maxwells Equations

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$$

$$\partial_\rho \epsilon^{\rho\sigma\mu\nu} F_{\mu\nu} = 0$$

Obeys $\partial_\mu J^\mu = 0$ current conservation

Lorentz Force law

$$m \frac{du^\mu}{d\tau} = q F^{\mu\nu} u_\nu$$

All of the above by the stationary action principle on $S[x^\mu(\sigma), A_\mu(x)]$

$x^\mu(\sigma) \rightarrow$ arbitrary path in spacetime

$A_\mu(x) \rightarrow$ arbitrary gauge field in spacetime

$$S = -mc \int d\sigma \left(\frac{dx^\mu}{d\sigma} \frac{dx_\mu}{d\sigma} \right)^{1/2}$$

$$- q \int d\sigma \, A_\mu(x(\sigma)) \frac{dx^\mu}{d\sigma}$$

$$-\frac{1}{4\mu_0}\int d^4x\; F^{\mu\nu} F_{\mu\nu}$$

EM in matter.

Charges and currents are separated into "bound" and "free".

"Bound" charges and currents average to ~~zero~~ zero because of rapid spatial oscillations.

They are characterized by a

$\vec{P} \rightarrow$ the polarization

$\simeq$ dipole moment per unit volume.

$\vec{M} \rightarrow$ the magnetization

$\simeq$ magnetic dipole moment per unit volume.

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$ → "displacement field"

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$$ → magnetic field

$\vec{B}$ → magnetic induction

Maxwell's Equations

$$\vec{\nabla} \cdot \vec{D} = \rho_{free}$$

$$\nabla \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \vec{J}_{free} + \frac{\partial \vec{D}}{\partial t}.$$

Linear response

$$\vec{P} = \epsilon_0 \chi \vec{E}$$

χ → electric susceptibility

So $\vec{D} = \epsilon \vec{E}$ with

$\epsilon = \epsilon_0(1+\chi)$ → permittivity

$K = \epsilon/\epsilon_0 = 1+\chi$ → dielectric constant

$$\vec{M} = \chi_m \vec{H} \qquad \chi_m \rightarrow \text{magnetic}$$

~~suspec~~ susceptibility

So $\vec{B} = \mu \vec{H}$

$$\mu = \mu_0 (1 + \chi_m) \rightarrow \text{magnetic permeability}$$

$$\vec{M} = \frac{1}{\mu_0} \frac{\chi_m}{1+\chi_m} \vec{B}$$

$$\chi_m > -1.$$

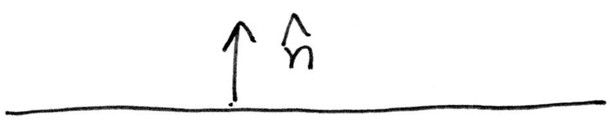
$0 > \chi_m > -1 \rightarrow$ diamagnet

$\chi_m > 0 \rightarrow$ paramagnet.

Reflection and Refraction of waves

Complicated results which follow from ~~applied~~ application of boundary conditions

$\epsilon_1 \; \mu_1$



$\epsilon_2 \; \mu_2$

$$\hat{n} \cdot (\vec{D}_1 - \vec{D}_2) = \sigma_{free}$$

surface charge density

$$\hat{n} \cdot (\vec{B}_1 - \vec{B}_2) = 0$$

$$\hat{n} \times (\vec{E}_1 - \vec{E}_2) = 0$$

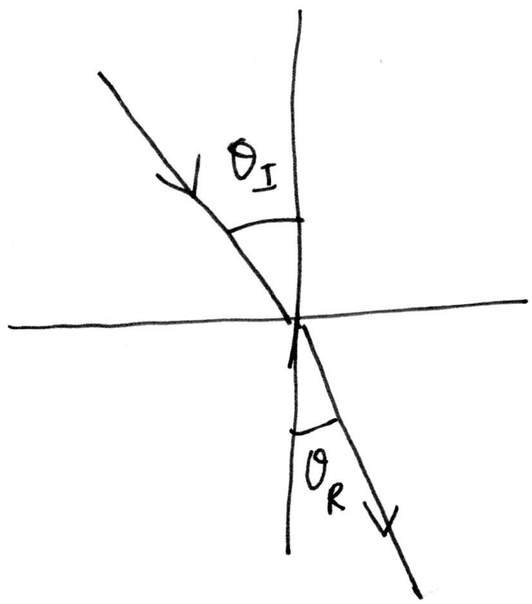
$$\hat{n} \times (\vec{H}_1 - \vec{H}_2) = \vec{K}$$

surface current density

Insulators: characterized by a refractive index

$$n = \frac{c}{v} \geq 1$$

where $v = \frac{1}{\sqrt{\mu \epsilon}}$ is the velocity of light in the insulator



$$n_1 \sin \theta_I = n_2 \sin \theta_R$$

Snell's Law.

Dispersion in Insulators

$$\vec{P}(\vec{r}, \omega) = \epsilon_0 \chi(\omega) \vec{E}(\vec{r}, \omega)$$

$$\chi(\omega) = \frac{\omega_p^2}{-\omega^2 + \omega_0^2 - i\gamma\omega}$$

$\omega_p^2 = \dfrac{n e^2}{\epsilon_0 m}$; $n \rightarrow$ density of bound electrons

$\omega_0 \rightarrow$ oscillation frequency of bound electrons

$\gamma \rightarrow$ damping constant of oscillations

$$\epsilon(\omega) = \epsilon_1 + i\,\epsilon_2 .$$

(i) $\epsilon_1 \gg \epsilon_2$; $\epsilon_1 > 0$.

<u>Transparent</u>

(ii) $\epsilon_2 \gg |\epsilon_1|$

<u>Absorption</u>

(iii) $\epsilon_1 < 0$, $|\epsilon_1| \gg \epsilon_2$

<u>Reflection</u>.

EM waves in metals

Conductivity of a metal, σ

$$\vec{J}_{free}(\omega) = \sigma(\omega)\,\vec{E}$$

$$\sigma(\omega) = \frac{\sigma_{DC}}{1 - i\omega\tau}$$

$\sigma_{DC} \rightarrow$ DC conductivity

$$= \frac{ne^2\tau}{m}\ ; \quad \tau \rightarrow \text{electron scattering time}$$

Can be absorbed into an effective dielectric constant

$$\epsilon(\omega) = \epsilon_0 + i\,\frac{\sigma(\omega)}{\omega}$$

$$\cancel{= \epsilon_0\left(1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2}\right)}$$

$$= \epsilon_0\left(1 - \frac{\omega_p^2}{\omega^2\left(1 + i/\omega\tau\right)}\right)$$

with $\boxed{\omega_p^2 = \frac{ne^2}{\epsilon_0 m}}$

3 regimes

(A) $\omega\tau \ll 1$

$$\epsilon(\omega) = \epsilon_0\left(1 + \frac{i\,\omega_p^2\tau}{\omega}\right)$$

$$\epsilon_2 \gg \epsilon_1$$

<u>Resonant absorption</u>.

EM waves penetrate a distance of the skin depth

$$\delta = \sqrt{\frac{2}{\mu_0\,\omega\,\sigma_{DC}}}$$

(B) $\omega\tau \gg 1$; $\omega < \omega_p$

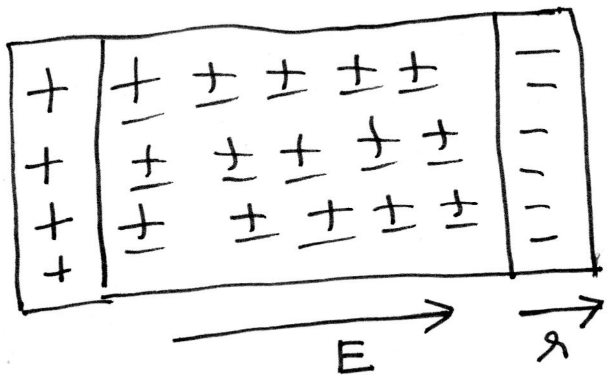
$$\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$$

<u>Total Reflection</u>

(C) $\omega\tau \gg 1$; $\omega > \omega_p$

<u>Transparent</u>

There are also longitudinal oscillations of charge density and electric fields at the plasma frequency



Optical Fibers

Like bound state ~~b~~ problems in quantum mechanics.

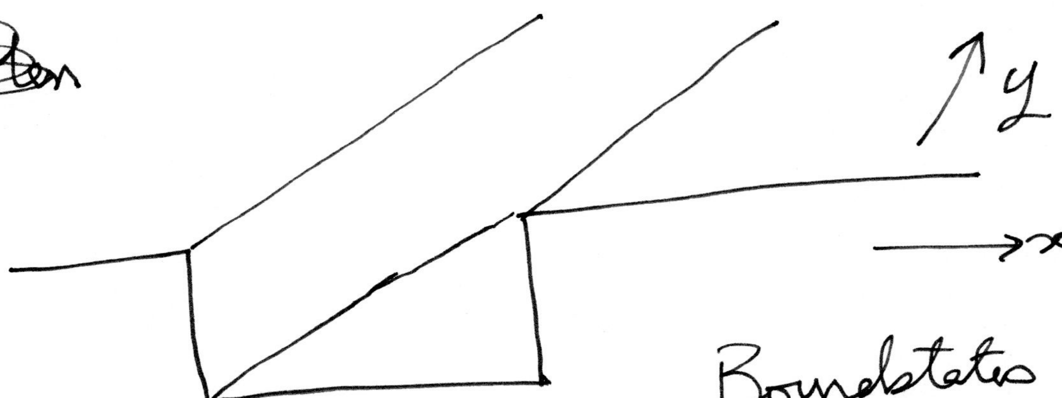
~~St~~ Schrodinger Equation

$$\left(\nabla^2 + \frac{2m}{\hbar^2}\left(E - V(\vec{x})\right)\right)\psi = 0$$

Maxwell's Equation

$$\left(\nabla^2 + n^2(\vec{x})\frac{\omega^2}{c^2}\right)\vec{E} = 0.$$

~~Poten~~



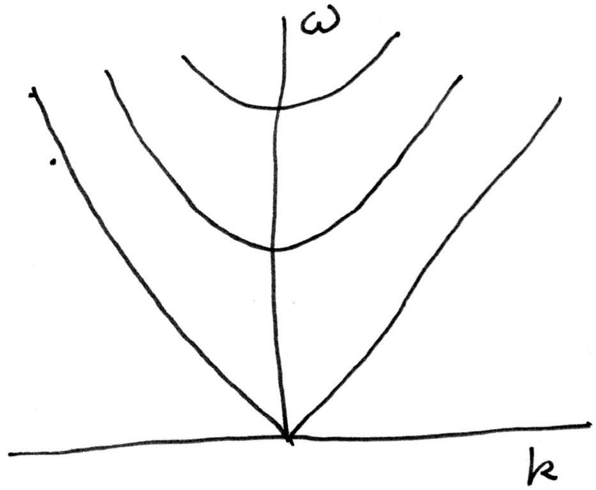
Boundstates only in transverse direction

$$\psi(x,y) = e^{iky}\phi(x)$$

$$\left(\nabla^2 + \frac{2m}{\hbar^2}\left(E - V(x)\right) - k^2\right)\phi = 0.$$

Modes of a planar Dielectric waveguide

Either TE or TM.



Photonic Lattices

Similar, with $n(x)$ periodic in one or more directions

Bloch's theorem

If $n(x+a) = n(x)$

then $E(x+a) = e^{ika} E(x)$

$k \rightarrow$ crystal wavevector

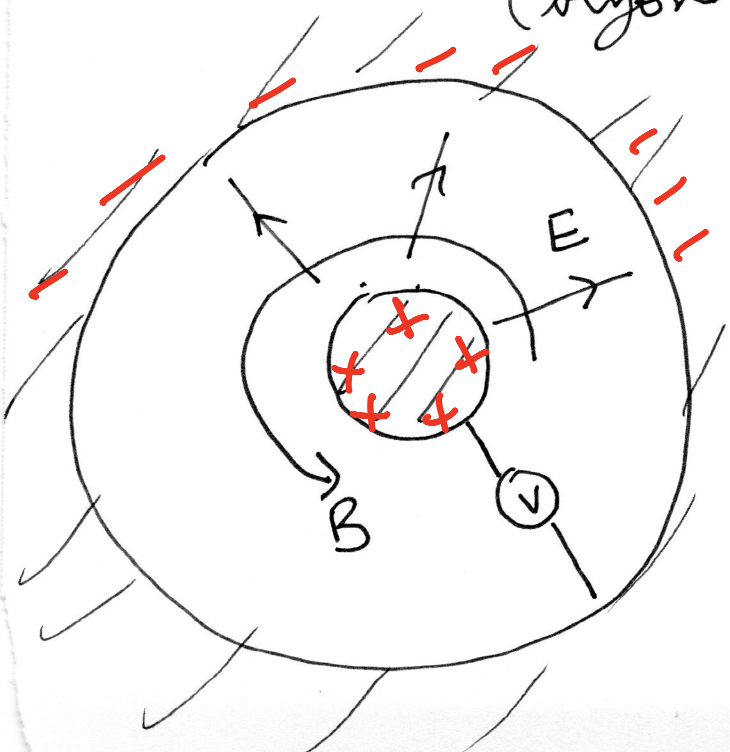
$$-\pi/a < k < \pi/a.$$

Frequency eigenmodes $\omega_n(k)$

There can be band gaps, where no light can propagate (but is instead reflected).

<u>Coaxial cables (and waveguides)</u>

Boundary conditions
$\vec{E} = \vec{B} = 0$ inside metal (beyond skin depth)



$\vec{E} \sim \cos(kz - \omega t)$

$\vec{B} \sim \cos(kz - \omega t)$

$\omega = ck.$

Telegraph Equations

$$\frac{\partial V}{\partial z} = -L\frac{\partial I}{\partial t}$$

$$\frac{\partial I}{\partial z} = -C\frac{\partial V}{\partial t}$$

$$LC = \mu\epsilon = \frac{1}{c^2}$$

Geometric Optics

Like WKB

$$\vec{E}(r,t) = E_0\, e^{i\omega S/c}\, e^{-i\omega t}$$

Maxwell's Equations

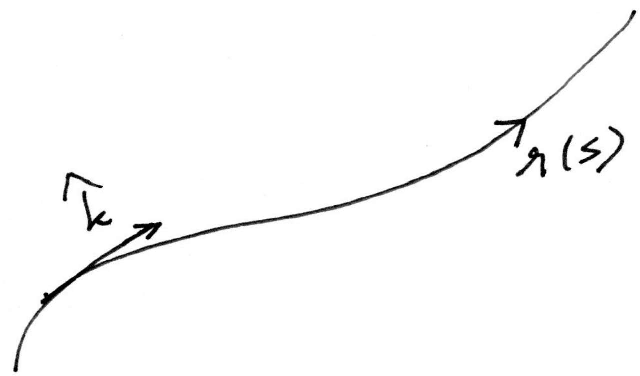
$$(\vec{\nabla}S)^2 = n^2(r)$$

$$\vec{\nabla}S = n(r)\,\hat{k}$$

$\hat{k} \rightarrow$ unit vector in direction of ray

Ray trajectory $\vec{r}(s)$

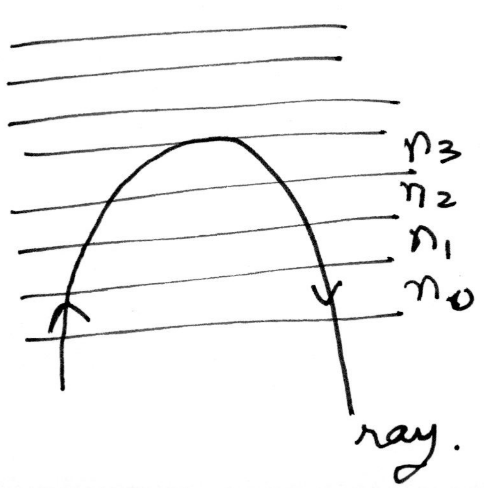
Defined $\frac{d\vec{r}}{ds} = \hat{k}$



Obeys ray equation

$$\frac{d}{ds}\left[n(r)\,\frac{d\vec{r}}{ds}\right] = \vec{\nabla} n(r)$$

Generalization of Snell's Law.



$n_0 > n_1 > n_2 > n_3$ ...

Scattering

$$\vec{E}(r,t) = E_0\left(\hat{e}_0\, e^{i\vec{k}_0\cdot\vec{r}} + \frac{e^{ikr}}{r}\vec{f}(k)\right]$$

$$|\vec{r}| \to \infty$$

Scattering cross section

$$\left.\frac{d\sigma}{d\Omega}\right| = \left|\epsilon^{*}\cdot\vec{f}(k)\right|^2$$

Polarization $\vec{\epsilon}$

Thomson Scattering off an electron

$$\left.\frac{d\sigma}{d\Omega}\right|_{\vec{e}_s} = r_e^2\,\left|\vec{e}_s\cdot\vec{e}_0\right|^2$$

$$r_e = \frac{e^2}{4\pi\epsilon_0 m c^2}$$

Rayleigh Scattering off a dielectric sphere

$$\vec{p} = \alpha \, \epsilon_0 \vec{E}$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{k^2 \alpha}{4\pi}\right)^2 |\vec{e}_s \cdot \vec{e}_0|^2$$