Conservation of Energy (Section 15.4)

$$\vec{F}_q = q(\vec{E} + \vec{v} \times \vec{B})$$

Work done on the particle $\sim \vec{F}_q \cdot dx = q\vec{E} \cdot \vec{dx}$.

The ~~[illegible]~~ magnetic field does no work.

$W_{mech} \rightarrow$ mechanical energy of particles with density $\rho(r)$ and current $\vec{J}(r) = \rho(r)\vec{v}(r)$

$$\frac{dW_{mech}}{dt} = \int d^3r\, \rho(\vec{r})\, \vec{E} \cdot \vec{v}(r)$$

$$= \int d^3r\, \vec{J}(r) \cdot \vec{E}(r)$$

$$= \int d^3r \left[\frac{1}{\mu_0}\vec{\nabla} \times \vec{B} - \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right] \cdot \vec{E}$$

Now use

$$\nabla \cdot (\vec{E} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{E}) - \vec{E} \cdot (\vec{\nabla} \times \vec{B})$$

$$= -\vec{B} \cdot \frac{\partial \vec{B}}{\partial t} - \vec{E} \cdot (\nabla \times \vec{B})$$

So

$$\int d^3r \frac{\partial}{\partial t}\left[\frac{1}{2}\epsilon_0 \vec{E}^2 + \frac{1}{2\mu_0}\vec{B}^2\right]$$

$$= -\int d^3r\, \vec{J} \cdot \vec{E} - \int_V d^3r \frac{1}{\mu_0} \nabla \cdot (\vec{E} \times \vec{B})$$

$U_{EM} \equiv$ energy of the electromagnetic field

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

the Poynting vector

= the energy current density.

$$\frac{\partial u_{EM}}{\partial t} + \vec{\nabla} \cdot \vec{S} = - \vec{J} \cdot \vec{E}$$

$$\Leftrightarrow \boxed{\frac{d U_{tot}}{dt} = \frac{d U_{EM}}{dt} + \frac{d U_{mech}}{dt} = - \int dA \, \vec{S} \cdot \vec{n}}$$

Compare to

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

Conservation of Momentum

$$\vec{F}_{mechanical} = \frac{d \vec{P}_{mechanical}}{dt} = \int d^3x \, \underbrace{\left(\rho \vec{E} + \vec{J} \times \vec{B} \right)}_{\vec{f}_{mech}}$$

$\neq 0$ in general

Some momentum must be carried by the electromagnetic field

Express ρ and $\vec{J}$ in terms of the $\vec{E}$ and $\vec{B}$ fields

$$\vec{F}_{mech} = \int d^3x \left\{ - \epsilon_0 \frac{\partial}{\partial t} (\vec{E} \times \vec{B}) + \epsilon_0 \left[(\vec{\nabla} \cdot \vec{E}) \vec{E} - \vec{E} \times (\nabla \times \vec{E}) \right] + \frac{1}{\mu_0} \left[(\nabla \cdot B) B - \vec{B} \times (\vec{\nabla} \times \vec{B}) \right] \right\}$$

$$= \int d^3x \left\{ - \frac{1}{c^2} \frac{\partial \vec{S}}{\partial t} + \epsilon_0 \left[(\vec{\nabla} \cdot \vec{E}) \vec{E} + (\vec{E} \cdot \vec{\nabla}) \vec{E} - \frac{1}{2} \vec{\nabla} (\vec{E} \cdot \vec{E}) \right] \right.$$

$$+\frac{1}{\mu_0}\left[(\nabla\cdot B)\vec{B} + (\vec{B}\cdot\vec{\nabla})B - \frac{1}{2}\vec{\nabla}(B\cdot B)\right]$$

Introduce momentum current density

$$T_{ij} = \epsilon_0\left[E_iE_j + c^2B_iB_j - \frac{\delta_{ij}}{2}\left[\vec{E}^2 + c^2\vec{B}^2\right]\right]$$

Then.

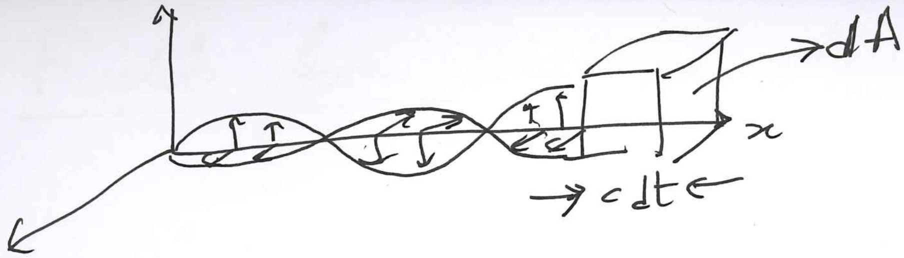
$$\vec{F}_{mech,i} = \frac{dP_{mech,j}}{dt} = \int d^3x \left\{-\frac{1}{c^2}\frac{\partial \vec{S}_j}{\partial t} + \frac{\partial}{\partial x_i}T_{ij}\right\}$$

$$\frac{d\vec{P}_{mech,j}}{dt} + \frac{d\vec{P}_{EM,j}}{dt} = \int_S d\vec{A}\; n_i T_{ij}$$

$$\vec{P}_{EM} = \int d^3x\, \vec{g}$$

$\vec{g} \to$ momentum density

$$\boxed{\vec{g} = \frac{\vec{S}}{c^2}} = \epsilon_0(\vec{E}\times\vec{B}).$$



Energy density in little region

$$\Delta E = S\, dt\, dA = \frac{S}{c}(c\, dt\, dA)$$
$$= c\, g\, dV$$

$\Delta p \rightarrow$ momentum in little region

So for light

$$\Delta E = c\, \Delta p$$

<u>For photons</u>

$$E = \hbar \omega \quad \text{(Planck)}$$
$$= \hbar c k$$

Momentum

$$p = \hbar k \quad \text{(de Broglie)}$$

So $E = c p$

Densities	Currents
Charge density $\rho(\vec{r})$	Current density $\vec{J}(r) = \rho(\vec{r})\,\vec{v}(r)$
EM Energy density $u_{EM}(\vec{r}) = \frac{1}{2}\epsilon_0 \vec{E}^2 + \frac{1}{2\mu_0}\vec{B}^2$	Energy current density $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$
EM Momentum density $\vec{g} = \vec{S}/c^2 = \epsilon_0(\vec{E}\times\vec{B})$	Momentum current density T_{ij}

$\vec{j}$ the component of momentum flowing in the i^{th} direction

$$T_{ij} = \epsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - \frac{\delta_{ij}}{2}\left(\epsilon_0 \vec{E}^2 + \frac{\vec{B}^2}{\mu_0}\right)$$

Conservation Laws

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$$

$$\frac{\partial u_{EM}}{\partial t} + \vec{\nabla}\cdot\vec{S} = -\vec{J}\cdot\vec{E}$$

$$\frac{\partial g_j}{\partial t} - \frac{\partial}{\partial x_i} T_{ij} = -f_{mech}$$