Fields from localized charges and currents

Electrostatics (4.1 and 4.2)

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r' \, \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

Assume $|\vec{r}'| \ll |\vec{r}|$

Then the Taylor series yields

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{|r|} - r'_j \partial_j\left(\frac{1}{|r|}\right)$$

$$+ \frac{1}{2}\left(r'_j \partial_j\right)\left(r'_k \partial_k\right)\left(\frac{1}{|r|}\right)$$

$$+ \cdots$$

First term

$$\Phi_0(\vec{r}) = \frac{Q}{4\pi\epsilon_0 r} \quad \text{where} \quad Q = \int d^3r' \, \rho(\vec{r}')$$

Second term

$$\Phi_1(r) = \frac{\vec{p}\cdot\vec{r}}{4\pi\epsilon_0 r^3} \quad \text{where} \quad \vec{p} = \int d^3r \, \rho(r)\,\vec{r}$$

is the dipole moment.

Note: $\vec{p}$ is independent of the origin if $Q = 0$.

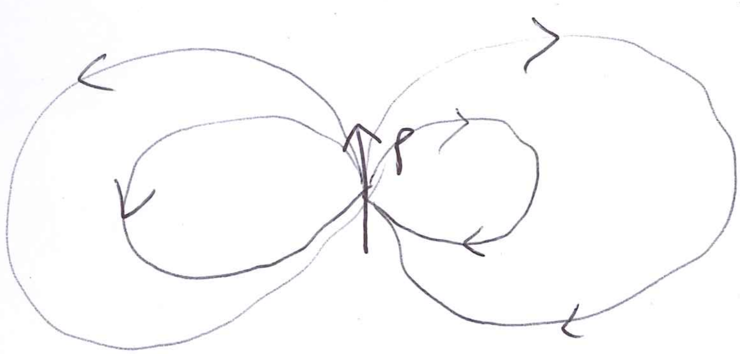
Dipolar field

$$\vec{E}(\vec{r}) = -\nabla \Phi_1(\vec{r})$$

$$= \frac{1}{4\pi\epsilon_0} \frac{3\hat{r}(\hat{r}\cdot\vec{p}) - \vec{p}}{r^3}$$

For $\vec{p} = p\hat{z}$ along the z direction

$$\vec{E} = \frac{p}{4\pi\epsilon_0 r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]$$



Third term

$$\Phi_2(r) = \frac{Q_{ij}}{4\pi\epsilon_0 r^5}\left(3r_i r_j - r^2\delta_{ij}\right)$$

where

$$Q_{ij} = \frac{1}{2}\int d^3r\, \rho(r)\, r_i r_j$$

is the quadrupole moment.

Note, we can take Q_{ij} traceless

$$Q_{ij} \longrightarrow Q_{ij} - \frac{1}{3}\delta_{ij} Q_{kk}$$

So

$$Q_{ij} = \frac{1}{2}\int d^3r \, \rho(\vec{r})\left(r_i r_j - \frac{\delta_{ij}}{3} r^2\right)$$

This does <u>not</u> change $\Phi_2(r)$.

Q_{ij} has 5 independent components.

Q_{ij} is independent of the origin of co-ordinates if $\vec{p}=0$ and $Q=0$.

<u>Magnetostatics</u> (11.1, 11.2)

$$A(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

As in electrostatics, assume $|\vec{r}| \gg |\vec{r}\,'|$ and

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{|\vec{r}|} + \frac{\vec{r}\,'\cdot\vec{r}}{|\vec{r}|^3}$$

First term is ~~per~~ proportional to $\int d^3r \, \vec{J}(\vec{r})$. But this vanishes

Note

$$\partial_j(r_k J_j) = \delta_{jk} J_j + r_k \partial_j J_j = J_k$$

But $\int d^3r\, \partial_j (r_k J_j) = 0$ by Gauss's law for localized currents

so $\int d^3r\, \vec{J} = 0$.

The first non-zero term is

$$A_k(\vec{r}) = \frac{\mu_0}{4\pi} \left[\int d^3r'\, J_k(r')\, r'_l \right] \frac{r_l}{r^3}$$

We will now show that this equals

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m}_0 \times \vec{r}}{r^3}$$

where $\boxed{\vec{m} = \frac{1}{2} \int d^3r\, \vec{r} \times \vec{J}(\vec{r})}$ is the magnetic moment.

We use

$$\partial_j (r_l r_k J_j) = r_l J_k + r_k J_l$$

and $\epsilon_{lki} (\vec{r} \times J)_i = r_l J_k - r_k J_l$

so $r_l J_k = \frac{1}{2} \epsilon_{lki} (\vec{r} \times \vec{J})_i + \frac{1}{2} \partial_j (r_l r_k J_j)$

The second term is a total derivative and integrates to 0.

The B field of a magnetic moment is

$$\vec{B} = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \left[\frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{r}}{r^3} \right]$$

$$= \frac{\mu_0}{4\pi} \left[\vec{m} \left(\nabla \cdot \frac{\vec{r}}{r^3} \right) - \left(\vec{m} \cdot \vec{\nabla} \right) \left(\frac{\vec{r}}{r^3} \right) \right]$$

$$= \frac{\mu_0}{4\pi} \frac{3(\hat{r} \cdot \vec{m})\,\hat{r} - \vec{m}}{r^3}$$

+ delta function at origin
which we ignore because
$|r| \gg$ size of moment.

This has the same form as the electric field from ~~near~~ an electric dipole $\vec{p}$.

Radiation and time-dependent sources

We will solve

$$\Box \vec{A} = \mu_0 \vec{J}$$

for a localized time-dependent source $\vec{J}(\vec{x},t)$.

We determine the Green's function

$$\left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(x,x';t,t') = \delta(\vec{x}-\vec{x}')\delta(t-t').$$

subject to suitable boundary conditions at ∞.

Then

$$\vec{A}(x,t) = \mu_0 \int d^3x'\, dt'\; G(x,x';t,t')\, \vec{J}(x',t').$$

$$\left\{\begin{array}{l}
\text{This is analogous to the electrostatic case, where} \\
-\nabla^2 G(x,x') = \delta(\vec{x}-x') \\
\Phi(\vec{x}) = \int d^3x'\; G(x,x')\, \dfrac{\rho(\vec{x}')}{\epsilon_0} \\
\text{and } G(x,x') = \dfrac{1}{4\pi\epsilon_0}\dfrac{1}{|\vec{x}-\vec{x}'|}
\end{array}\right.$$

§ Review of Fourier analysis

1D
$$f(x) = \int_{-\infty}^{\infty} f_k \, e^{ikx} \frac{dk}{2\pi} \qquad (1)$$

$$f_k = \int_{-\infty}^{\infty} dx \, f(x) \, e^{-ikx} \qquad (2)$$

Substituting (2) in (1)

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \int_{-\infty}^{\infty} dx' f(x') e^{-ikx'}$$

$$= \int_{-\infty}^{\infty} dx' f(x') \underbrace{\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-x')}}_{\delta(x-x')}$$

$$\boxed{\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-x')} = \delta(x-x')}$$

In space time

$$G(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^3} \frac{d\omega}{2\pi} G(k, \omega) \, e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

$$G(k, \omega) = \int d^3\vec{x} \, dt \, G(\vec{x}, t) \, e^{-i(\vec{k}\cdot\vec{x} - \omega t)}$$

We write

$$G(\vec{x}, \vec{x}'; t, t') = \int \frac{d^3k}{(2\pi)^3} \frac{d\omega}{2\pi} G(k, \omega) \times e^{i[k\cdot(x - x') - \omega(t - t')]}$$

Substituting this in the eqn for the Green's fn we obtain

$$\int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3}\frac{d\omega}{(2\pi)}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')-i\omega(t-t')} \times \left[\left(k^2-\omega^2/c^2\right)G(k,\omega) = 1\right]$$

so $$\boxed{G(\vec{k},\omega) = \frac{1}{k^2-\omega^2/c^2}}$$

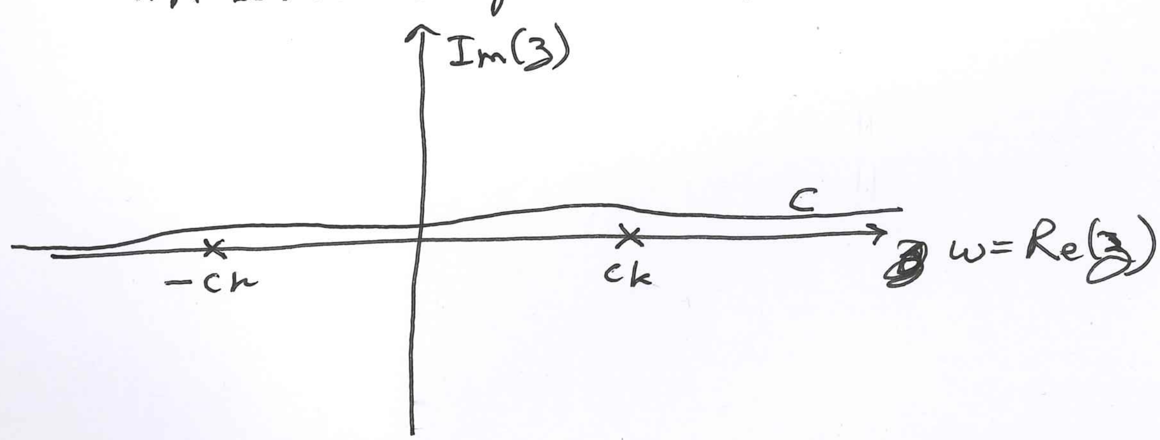
and

$$G(\vec{r},\vec{r}';t,t') = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{R}} \times \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega\tau}\, \frac{1}{k^2-\omega^2/c^2}$$

where $\tau \equiv t-t'$, $\vec{R} = \vec{r}-\vec{r}'$

Q: <u>How do we deal with the pole at $\omega = \pm ck$?</u>

More generally, we can pick any contour in the complex ω plane

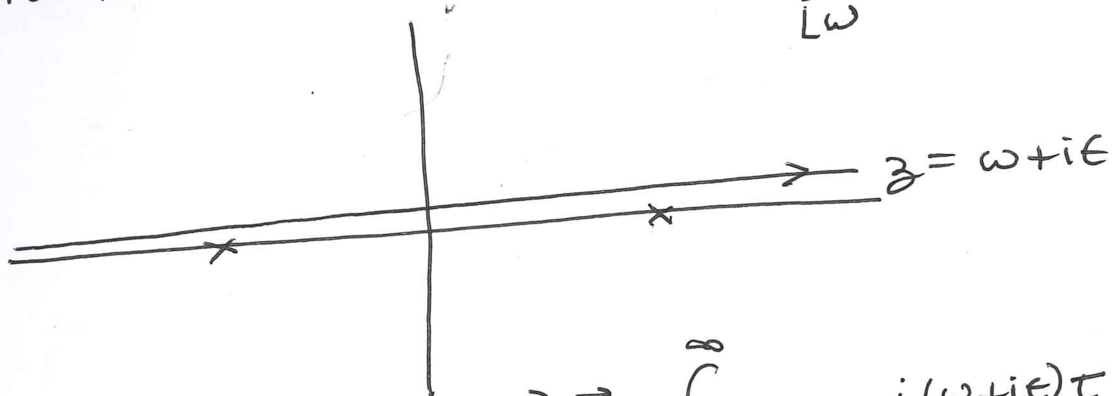


$$G_C(\vec{R},\tau) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{R}} \int_C \frac{dz}{2\pi} e^{-iz\tau} \frac{1}{k^2 - z^2/c^2}$$

The choice of C corresponds to the choice of boundary conditions. We choose

$$z = \omega + i\epsilon, \quad \epsilon > 0, \quad \epsilon \to 0.$$

This corresponds to "causal" boundary conditions and leads to the "retarded" Green's functions.

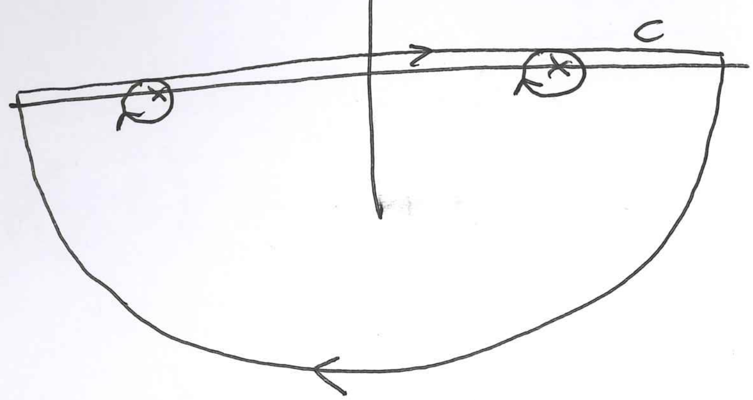


$$G_\epsilon(R,\tau) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{R}} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i(\omega+i\epsilon)\tau} \cdot \frac{1}{k^2 - z^2/c^2}$$

For $\tau < 0$, we can close the contour in the upper half plane and

$$G_\epsilon(R, \tau < 0) = 0.$$

For $\tau > 0$, we pick up contributions from the 2 poles



$$\oint_C dz\, e^{-iz\tau} \frac{1}{k^2 - z^2/c^2}$$

$$= \frac{2\pi c}{k} \sin(ck\tau)$$

So

$$G_\epsilon(\vec{R}, \tau > 0)$$

$$= \int \frac{d^3k}{8\pi^3} e^{i\vec{k}\cdot\vec{R}} \frac{c}{k} \sin(ck\tau)$$

$$= \frac{c}{2\pi^2 R} \int_0^\infty dk \sin(kR) \sin(ck\tau)$$

after angular integral over $\vec{k}$

$$= \frac{c}{4\pi^2 R} \int_{-\infty}^{\infty} dk \sin(kR) \sin(ck\tau)$$

$$= \frac{c}{4\pi R} \left(\delta(R - c\tau) - \delta(R + c\tau) \right).$$

Because $\tau > 0$, the second δ function vanishes.

So

$$\boxed{G_{\epsilon}(\vec{R}, \tau) = \frac{\delta(t - t' - |r - r'|/c)}{4\pi |\vec{r} - \vec{r}'|}}$$

Just like electrostatics, but with a delta function in time with a time delay of $|\vec{r} - \vec{r}'|/c$.

So

$\cancel{\vec{J}(\vec{r}, t)}$

$$\boxed{\vec{A}(\vec{r}, t) = \mu_0 \int d^3r' \frac{J(\vec{r}', t - |r - r'|/c)}{4\pi |\vec{r} - \vec{r}'|}}$$

We will usually look at

$$\vec{J}(\vec{r},t) = \vec{J}(\vec{r})\, e^{-i\omega t}.$$

Then we can also assume

$$\vec{A}(\vec{r},t) = \vec{A}(\vec{r})\, e^{-i\omega t}$$

and the equation is

$$\left(-\nabla^2 - \omega^2/c^2\right)\vec{A} = \mu_0 \vec{J}$$

The Hemholtz eqn.

We define the Green's function by

$$\left(-\nabla^2 - \omega^2/c^2\right) G_\omega(\vec{r}-\vec{r}\,') = \delta(\vec{r}-\vec{r}\,').$$

The previous solution corresponds to

$$\boxed{G_\omega(\vec{r}-\vec{r}\,') = \frac{e^{i\omega|\vec{r}-\vec{r}\,'|/c}}{4\pi\,|\vec{r}-\vec{r}\,'|}}$$

"outgoing waves"

$$\vec{A}(\vec{r}) = \mu_0 \int d^3r'\, G_\omega(\vec{r}-\vec{r}\,')\,\vec{J}(\vec{r}\,')$$