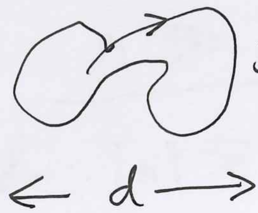
Radiation from a localized source



$J(\vec{r}') \times e^{-i\omega t}$

$k = \omega/c \,, \quad \lambda = \dfrac{2\pi}{k}$

Assume $\lambda \gg d$ so details of source don't matter

$$\vec{A}(\vec{r}, t) = \frac{\mu_0 e^{-i\omega t}}{4\pi} \int d^3 r' \, \frac{e^{ikR} J(r')}{R}$$

where $R = |r - r'|$.

Near field

$$d < r \ll \lambda.$$

$$R = |r - r'| \ll \lambda$$

$$\Rightarrow kR \ll 1$$

so

$$\vec{A}(\vec{r}, t) = \frac{\mu_0 e^{-i\omega t}}{4\pi} \int \frac{d^3 r' \, J(r')}{|r - r'|}$$

Same as <u>magnetostatics</u>!

Response is instantaneous because c is very large.

Far field

$$r \gg \lambda \gg d.$$

so $|r| \gg |\vec{r}\,'|$ and we can write

$$|\vec{r} - \vec{r}\,'| \approx r\sqrt{1 - 2\frac{\vec{r}\cdot\vec{r}\,'}{r^2}} \qquad (r \equiv |\vec{r}|)$$

$$\approx r - \frac{\vec{r}\cdot\vec{r}\,'}{r}$$

$$= r - \hat{r}\cdot\vec{r}\,'.$$

so

$$\vec{A}(\vec{r},t) = \frac{\mu_0 e^{-i\omega t}}{4\pi}\int d^3r'\,\frac{e^{ik(r - \hat{r}\cdot\vec{r}\,')}\vec{J}(r')}{r - \hat{r}\cdot\vec{r}\,'}$$

$$\approx \frac{\mu_0 e^{-i\omega t}}{4\pi r}\int d^3r'\,\vec{J}(r')\,e^{ikr}\Big[1 \underbrace{- ik\hat{r}\cdot\vec{r}\,' + \frac{\hat{r}\cdot\vec{r}\,'}{r}}_{\substack{\text{both terms}\\ \text{are small.}}} + \dots\Big].$$

Ignore for now.

DIPOLE TERM

$$\vec{A}(\vec{r},t) = \frac{\mu_0 e^{i(kr - \omega t)}}{4\pi r}\int d^3r'\,\vec{J}(\vec{r}\,')$$

As before use

$$\partial_j (r_i J_j) = J_i + r_i \partial_j J_j$$

$$= J_i + r_i \left(-\frac{\partial \rho}{\partial t}\right)$$

$$= J_i + i\omega r_i \rho$$

and integrate over space to obtain

$$\int d^3r' \vec{J}(\vec{r}') = -i\omega \int d^3\vec{r}' \, \vec{r}' \rho(\vec{r}')$$

$$= -i\omega \vec{p}$$

electric dipole moment !

So $$\boxed{A(\vec{r},t) = \frac{\mu_0 e^{+i(kr-\omega t)}}{4\pi r} (-i\omega \vec{p})}$$

$$\vec{\nabla} \times \vec{A} = \frac{\mu_0}{4\pi} \vec{\nabla}\left(\frac{e^{i(kr-\omega t)}}{r}\right) \times \vec{p} \, (-i\omega)$$

In spherical co-ordinates

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \text{angular terms}$$

So $$\boxed{\vec{B} = \frac{\mu_0}{4\pi} \frac{\omega^2}{c} (\hat{r} \times \vec{p}) \frac{e^{i(kr-\omega t)}}{r}}$$

Deduce $\vec{E}$ from Maxwells Eqn

$$\vec{\nabla} \times \vec{B} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$$

So $\vec{E} = \frac{c^2}{-i\omega} \vec{\nabla} \times \vec{B}$

$$\boxed{\vec{E} = -\frac{\mu_0}{4\pi} \omega^2 \left[\hat{r} \times (\hat{r} \times \vec{p})\right] \frac{e^{i(kr-\omega t)}}{r}}$$

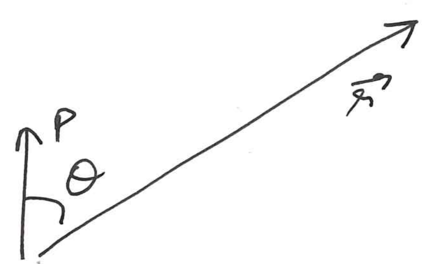
Poynting vector

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

At a fixed frequency

$$\vec{S} = \frac{1}{\mu_0} Re[\vec{E}] \times Re[\vec{B}]$$

$$\sim p^2 \hat{r} \frac{\sin^2\theta}{r^2} \cos^2(kr - \omega t)$$



Power radiated per solid angle

ELECTRIC DIPOLE RADIATION

$$\overline{\frac{dP}{d\Omega}} = \overline{r^2 \vec{S} \cdot \hat{r}}$$

time average

$$= \frac{ck^4}{32\pi^2\epsilon_0} p^2 \sin^2\theta$$

More generally, we need not assume $\lambda \gg d$. Then in the regime $r \gg d, \lambda$ we have

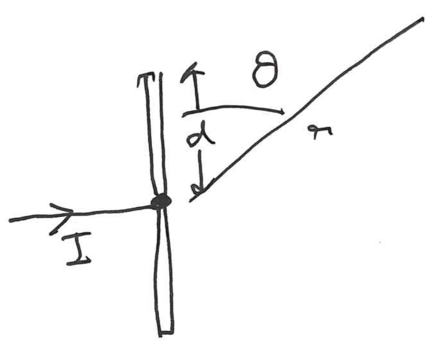
$$\vec{A}(r,t) = \frac{\mu_0}{4\pi} \frac{e^{i(kr-\omega t)}}{r} \int d^3r' \, \vec{J}(r') \times e^{-i\vec{k}\cdot\vec{r}'}$$

where $\vec{k} \equiv k\hat{r}$
$= \frac{\omega}{c}\hat{r}$.

$$\vec{B}(\vec{r}) = \frac{i\omega}{c}\hat{r} \times \vec{A}(\vec{r})$$

$$\vec{E}(\vec{r}) = -i\omega\, \hat{r} \times [\hat{r} \times \vec{A}(\vec{r})].$$

Dipole antenna



Current in wire is determined by condition that tangential electric field vanish at the surface.

$$I(z,t) = \mathrm{Re}\left[I_0 \sin\left(k(d-|z|)\right) e^{-i\omega t} \right]$$

Using expressions above we obtain

$$\vec{A}(\vec{r}) = \frac{\mu_0 I_0}{2\pi} \frac{e^{i(kr-\omega t)}}{kr} \times$$

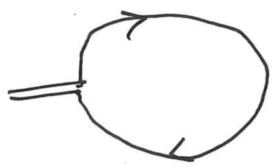
$$\left[\frac{\cos(kd\cos\theta) - \cos kd}{\sin^2\theta}\right]\hat{z}$$

Radiated Power

$$\frac{dP}{d\Omega} = \frac{\mu_0 c I_0^2}{8\pi^2}\left[\frac{\cos(kd\cos\theta) - \cos kd}{\sin\theta}\right]^2$$

Magnetic Dipole and Electric Quadrupole Radiation.

For some current distributions the electric dipole term vanishes e.g. a circular current



$$\int d^3r'\, \vec{J}(r') = 0.$$

For these cases we have to consider the next term in $\int d^3r'\, \vec{J}(\vec{r}')\, e^{-i\vec{k}\cdot\vec{r}'}$

$$\approx \int d^3r'\, \vec{J}(\vec{r}') - i \int d^3r'\, \vec{J}(\vec{r}')\, \vec{k}\cdot\vec{r}'.$$

We use the identity

$$(\vec{k}\cdot\vec{r})\vec{J} = \underbrace{\frac{1}{2}(\vec{r}\times\vec{J})\times\vec{k}}_{\text{magnetic dipole}} + \underbrace{\frac{1}{2}\left[(\vec{k}\cdot\vec{r})J + (\vec{k}\cdot\vec{J})\vec{r}\right]}_{\text{electric quadrupole}}.$$

So

$$\vec{A}_{\text{magnetic dipole}}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{e^{i(kr - \omega t)}}{r} \times ik\,(\hat{r} \times \vec{m})$$

and then

$$\vec{B}_{\text{magnetic dipole}} = \frac{\mu_0}{4\pi} k^2 \left[\hat{r} \times (\vec{m} \times \hat{r})\right] \frac{e^{i(kr - \omega t)}}{r}$$

$$\vec{E}_{\text{magnetic dipole}} = -\frac{\mu_0}{4\pi} \frac{\omega^2}{c} \left[\hat{r} \times \vec{m}\right] \frac{e^{i(kr - \omega t)}}{r}.$$

Note electric and magnetic dipole radiation map onto each other under

$$\vec{p} \longrightarrow \vec{m}/c$$

$$\vec{B} \longrightarrow -\vec{E}/c$$

$$\vec{E} \longrightarrow c\vec{B}.$$

Also power ~~radiatio~~ radiated

$$\frac{dP}{d\Omega} = \frac{\mu_0 c k^4}{32\pi^2} m^2 \sin^2\theta.$$

~~Also~~ Note

$$\frac{(\text{Power})_{\text{electric dipole}}}{(\text{Power})_{\text{magnetic dipole}}} \approx \left(\frac{c|\vec{p}|}{|\vec{m}|}\right)^2 \sim \left(\frac{c}{v}\right)^2 \gg 1.$$

where $\vec{m} = \frac{1}{2}\int d^3r \; \vec{r} \times \vec{J}(\vec{r})$

and $\vec{p} = \int d^3r \; \vec{r}\, \rho(\vec{r})$.

Finally, the Electric Quadrupole term:

$$\int d^3r \, J_i(\vec{r})\, e^{-i\vec{k}\cdot\vec{r}} = \cdots \int -\frac{i}{2}\left[k_\ell r_\ell J_i + k_\ell J_\ell r_i\right] d^3\vec{r}$$

Lets write

$$I_i = \int d^3r \left[\underbrace{k_\ell r_\ell J_i}_{= J_m \partial_m r_i} + \underbrace{k_\ell J_\ell r_i}_{= J_m \partial_m r_\ell}\right]$$

Integrate by parts and use

$$\partial_m J_m = i\omega\rho$$

Then

$$I_i = -i\omega \int d^3r \, \rho(r)\, k_\ell \left[r_i r_\ell + r_\ell r_i\right]$$

$$- \underbrace{\int d^3r \, k_\ell \left[J_\ell r_i + J_i r_\ell\right]}_{= I_i!}$$

So

$$I_i = -i\omega k_\ell \int d^3r\, \rho(\vec{r})\, r_\ell r_i$$

$$\equiv -2i\omega k_\ell Q_{\ell i}$$

So

$$\vec{A}_{\text{quadrupole},\, i}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{e^{i(kr - \omega t)}}{r} \times \left[-\omega k_\ell Q_{k \ell i}\right]$$

~~Reason~~ (Recall $Q_{\ell m} = \frac{1}{2}\int d^3r\, r_\ell r_m \rho(r)$ is the quadrupole moment)

From this we obtain the electric fields.

$$\vec{B} = -\frac{i\mu_0\omega^3}{4\pi c^2} \frac{(\hat{r} \times (\overleftrightarrow{Q} \cdot \hat{r}))}{r}$$

$$\vec{E} = \frac{i\mu_0\omega^3}{4\pi c r}\left[\left(\hat{r} \cdot \overleftrightarrow{Q} \cdot \hat{r}\right)\hat{r} - \overleftrightarrow{Q} \cdot \hat{r}\right].$$

and the power radiated

$$\frac{dP}{d\Omega} = \frac{\mu_0\omega^6}{16\pi^2 c^3}\left|\hat{r} \times \left(\overleftrightarrow{Q} \cdot \hat{r}\right)\right|^2$$