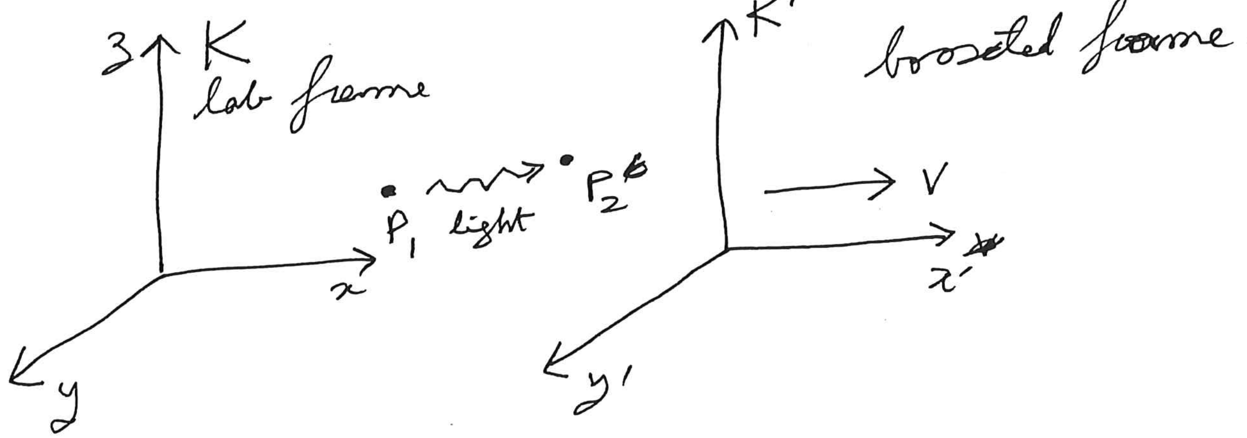
Relativity



A light beam is sent from $P_1 = (x_1, y_1, z_1)$
to $P_2 = (x_2\ y_2\ z_2)$

Velocity of light $= c$

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - c^2(t_2 - t_1)^2 = 0.$$

In moving frame

$$(x_2' - x_1')^2 + (y_2' - y_1')^2 + (z_2' - z_1')^2 - c^2(t_2' - t_1')^2 = 0$$

Consider infinitesimally separated events

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

Principle of relativity

$ds^2 = 0$ implies $ds'^2 = 0$ for all v.

By homogeneity of space and time

$$ds^2 = a(|v|)\, ds'^2$$

Now consider 3 frames K, K_1 (velocity $\vec{v}_1$)
K_2 (velocity $\vec{v}_2$)

$$ds^2 = a(|\vec{v}_1|)\, ds_1^2$$
$$ds^2 = a(|\vec{v}_2|)\, ds_2^2$$
$$ds_1^2 = a(|\vec{v}_2 - \vec{v}_1|)\, ds_2^2$$

Only consistent solution for all angles between $\vec{v}_1$ and $\vec{v}_2$ is $\underline{\underline{a = 1}}$

So we conclude

$ds^2 = ds'^2$ for all intervals

$$\boxed{\begin{aligned} &c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2 \\ &= c^2(t_2' - t_1') - (x_2' - x_1')^2 - (y_2' - y_1')^2 \\ &\qquad - (z_2' - z_1')^2 \end{aligned}}$$

for any 2 events at P_1, P_2

Principle of relativity

Lorentz transformation

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma & \frac{v}{c}\gamma & 0 & 0 \\ \frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$\boxed{\begin{aligned} ct &= \gamma ct' + \frac{v}{c}\gamma x' \\ x &= v\gamma t' + \gamma x' \end{aligned}}$$

$$\begin{aligned} (ct)^2 - x^2 &= \gamma^2 \Big(c^2 t'^2 + \frac{v^2}{c^2} x'^2 + 2vt'x' \\ &\qquad - v^2 t'^2 - x'^2 - 2vt'x' \Big) \\ &= \frac{(c^2 t'^2 - x'^2)(1 - v^2/c^2)}{1 - v^2/c^2} \quad ✓ \end{aligned}$$

Addition of velocities

$$c\,dt = \gamma c\,dt' + \frac{v}{c}\gamma\,dx'$$

$$dx = \gamma v\,dt' + \gamma\,dx'$$

$$u = \frac{dx}{dt} = \frac{\gamma v\,dt' + \gamma\,dx'}{\gamma\,dt' + \frac{v}{c^2}\gamma\,dx'}$$

$$= \frac{v + u'}{1 + \frac{vu'}{c^2}}$$

$$\vec{u}_{\parallel} = \frac{\vec{v} + \vec{u}'_{\parallel}}{1 + \frac{\vec{v}\cdot\vec{u}'}{c^2}}$$

Similarly for y and z

$$\vec{u}_{\perp} = \frac{u'_{\perp}}{\gamma\left(1 + \frac{\vec{v}\cdot\vec{u}'}{c^2}\right)}$$

Note: If $\vec{u}'_{\parallel} = c\hat{v}$ then

$\vec{u}_{\parallel} = c \rightarrow$ velocity of light is the same in all frames.

Four vector notation

$$x^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \qquad \mu = 0, 1, 2, 3.$$

a "contravariant" vector

We also introduce a metric tensor

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & 0 \\ & 0 & -1 & \\ & & & -1 \end{pmatrix}.$$

So

$$ds^2 = \sum_{\mu\nu=0}^{3} dx^\mu dx^\nu g_{\mu\nu}$$

will drop the summation and sum over repeated indices.

It is also useful to define

$$x_\mu \equiv g_{\mu\nu} x^\nu \qquad \text{a "covariant" vector.}$$

Then

$$ds^2 = dx^\mu dx_\mu$$

$\boxed{\text{Must always contract upper and lower indices to maintain Lorentz invariance}}$

Lorentz transformation

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

$$g_{\mu\nu} x'^\mu x'^\nu = g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma x^\rho x^\sigma$$

So we have

$$= g_{\rho\sigma} x^\rho x^\sigma$$

by Lorentz invariance

$$\boxed{\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma g_{\mu\nu} = g_{\rho\sigma}}$$

All Λ satisfying this are Lorentz transformation

We can also raise and lower indices on tensors.

$$x_\mu = g_{\mu\nu} x^\nu$$

We define $x^\nu = g^{\nu\mu} x_\mu$

$$\Rightarrow g^{\mu\nu} = (g^{-1})_{\mu\nu} = g_{\mu\nu}.$$

Also

$$g^\mu{}_\nu = g^{\mu\lambda} g_{\lambda\nu} = \delta^\mu{}_\nu.$$

So we have

$$\Lambda_{\nu\rho} \Lambda^{\nu}{}_{\sigma} = g_{\rho\sigma}$$

or $$\Lambda_{\nu}{}^{\rho} \Lambda^{\nu}{}_{\sigma} = \delta^{\rho}{}_{\sigma}$$

etc.

We can summarize this by

$$\Lambda^{T} \Lambda = \mathbb{1} \quad \text{or} \quad (\Lambda^{-1}) = \Lambda^{T}$$

$$\text{i.e.} \quad (\Lambda^{-1})^{\mu}{}_{\nu} = \Lambda_{\nu}{}^{\mu}$$

Just have to remember to contract upper and lower indices

<u>Important fact</u>

$\frac{\partial}{\partial x_{\mu}} \equiv \partial_{\mu}$ transforms like a 4-vector with a lower index.

Pf. $$dx'^{\mu} = \Lambda^{\mu}{}_{\nu} dx^{\nu} \quad \text{(Defines } \Lambda\text{)}$$

and $$\frac{\partial}{\partial x'^{\mu}} = \frac{\partial}{\partial x^{\nu}} \frac{\partial x^{\nu}}{\partial x'^{\mu}}$$

$$= \frac{\partial}{\partial x^{\nu}} (\Lambda^{-1})^{\nu}{}_{\mu}$$

$$= \Lambda_{\mu}{}^{\nu} \frac{\partial}{\partial x^{\nu}}$$ <u>as required.</u>

If A^μ is a 4-vector

$\partial_\mu A^\mu$ is a scalar

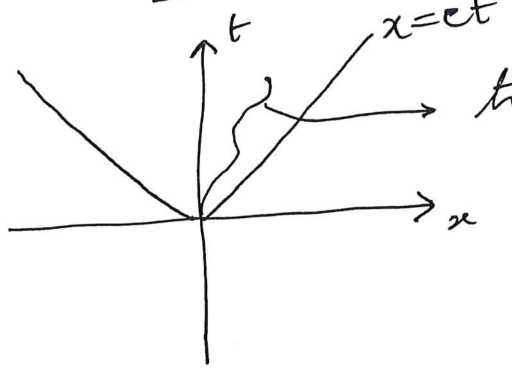
and

$$\partial_\mu \partial^\mu = \partial_\mu \partial_\nu g^{\mu\nu}$$

$$= c^2 \frac{\partial^2}{\partial t^2} - \nabla^2$$

is also a scalar

Four velocity



$d\tau \rightarrow$ ~~por~~ proper time ~~vector~~ interval

$$= ds/c$$

$$= \frac{1}{c}\sqrt{c^2 dt^2 - dx^2}$$

$$= dt\sqrt{1 - v^2/c^2}$$ is a Lorentz invariant.

Four velocity

$$u^\mu = \frac{dx^\mu}{d\tau} \rightarrow \text{a 4-vector}$$

Note $$u^\mu u_\mu = \frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau} = \frac{c^2 dt^2 - d\vec{x}^2}{dt^2 - \vec{v}^2/c^2}$$

$$= c^2.$$

Four momentum

$$p^\mu = m u^\mu = m \begin{pmatrix} c\gamma \\ \gamma v_x \\ \gamma v_y \\ \gamma v_z \end{pmatrix}$$

We identify this $\mathcal{D}$ with

$$p^\mu = \begin{pmatrix} \mathcal{E}/c \\ \vec{p} \end{pmatrix} \quad \text{where } \vec{p} = m\gamma\vec{v}$$

and where $\mathcal{E}$ is the relativistic energy

$$\mathcal{E} = mc^2\gamma = \frac{mc^2}{\sqrt{1 - v^2/c^2}} = mc^2 + \frac{1}{2}mv^2 + \dots$$

Electrodynamics

Total charge $Q = \int d^3r \, \rho(r)$

is a Lorentz scalar.

$\rightarrow$ Experimental fact $\rightarrow$ atoms are neutral in all frames of reference.

Also
$c\,dt\,d^3r$ is a Lorentz scalar

$$\longrightarrow c\,dt'\,d^3r'\,\underbrace{|\det\Lambda|}_{=1}.$$

So $\rho(\vec{r})$ must ~~be~~ transform like the time component of a Lorentz scalar.

We introduce the 4- current

$$J^\mu = \begin{pmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{pmatrix}$$

where recall

$$\rho(\vec{r}) = \sum_k q_k \, \delta^3(\vec{r} - \vec{r}_k(t))$$

$$\vec{J}(\vec{r}) = \sum_k q_k \frac{d\vec{r}_k}{dt} \delta^3(\vec{r} - \vec{r}_k(t))$$

Then the continuity equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} . \vec{J} = 0$$

has the Lorentz invariant form

$$\boxed{\partial_\mu J^\mu = 0}.$$

Note: we can write

$$\boxed{J^\mu = \int d\tau \sum_k q_k \frac{dx_k^\mu}{d\tau} \delta^{(4)}(x^\mu - x_k^\mu(\tau))}$$

Maxwell's Equations.

Use Lorentz gauge

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad ; \quad \vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}.$$

with $$\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0.$$

So we introduce the 4-vector

$$A^\mu = \begin{pmatrix} \Phi/c \\ \vec{A} \end{pmatrix}.$$

Then Lorentz gauge condition is

$$\boxed{\partial_\mu A^\mu = 0.}$$

Maxwell Equations were

$$\Box \Phi = \rho/\epsilon_0 \quad , \quad \Box \vec{A} = \mu_0 \vec{J}$$

These can be rewritten as

$$\boxed{\partial_\mu \partial^\mu A^\nu = \mu_0 J^\mu}$$

What about the $\vec{E}$ and $\vec{B}$ fields?

Introduce

$$F^{\mu\nu} = -F^{\nu\mu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

Note $F^{00} = F^{ii} = 0$.

$$F^{12} = \partial^x A^y - \partial^y A^x$$

$$= -\partial_x A^y + \partial_y A^x = -B^z$$

$$F^{23} = -B^x$$

$$F^{31} = -B^y$$

$$F^{01} = \partial^0 A^x - \partial^x A^0$$

$$= \frac{1}{c}\frac{\partial A_x}{\partial t} + \frac{1}{c}\frac{\partial \Phi}{\partial t} = -\frac{E_x}{c} \quad \text{etc.}$$

So

$$F^{\mu\nu} = \begin{pmatrix} 0 & -B_z & & \\ & 0 & -B_x & \\ & & 0 & -B_y \\ & & & 0 \end{pmatrix}$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

The electromagnetic field strength is a second-rank antisymmetric Lorentz tensor.

$$\boxed{\begin{array}{c} \text{Maxwell's Equations} \\ \partial_\mu F^{\mu\nu} = \mu_0 J^\nu \\ \partial_\rho \epsilon^{\rho\sigma\mu\nu} F_{\mu\nu} = 0 \end{array}}$$

where $\epsilon^{\rho\sigma\mu\nu}$ is the fully antisymmetric 4-th rank unit tensor

$$\epsilon^{0123} = 1 = -\epsilon^{1023} = \epsilon^{1032} = \cdots$$

Check by explicit substitution.

Also by comparing with the non-relativistic limit

$$m\frac{d\vec{v}}{dt} = q\left(\vec{E} + \vec{v}\times\vec{B}\right)$$

we can guess the relativistic form of the Lorentz force law.

$$\boxed{m\frac{du^\mu}{d\tau} = qF^{\mu\nu}u_\nu} \quad (\star)$$

Recall $p^\mu = m u^\mu = \begin{pmatrix} mc\gamma \\ \vec{p} \end{pmatrix} = \begin{pmatrix} mc\gamma \\ m\vec{v}\gamma \end{pmatrix}$.

So spatial components of $(\star)$ yield

$$\frac{d\vec{p}}{d\tau} = \gamma\frac{d\vec{p}}{dt} = q\gamma\left(\vec{E} + \vec{v}\times\vec{B}\right)$$

which yields

$$\boxed{m\frac{d}{dt}\left[\frac{\vec{v}}{\sqrt{1-\vec{v}^2/c^2}}\right] = q\left(\vec{E} + \vec{v}\times\vec{B}\right)}$$

From the ~~spatial~~ time component of (*)

recalling

$$p^{\mu} = \begin{pmatrix} \mathcal{E}/c \\ \vec{p} \end{pmatrix}$$

we obtain)

$$\frac{d\mathcal{E}}{dt} = q\,\vec{v}\cdot\vec{E}$$

i.e. rate of change of energy = work done by external field.

$\Rightarrow$ Our identification of $p^0 = \mathcal{E}/c$ was correct.

~~Also on~~ Also note

$$\frac{d}{d\tau}\left(u^{\mu}u_{\mu}\right) = 2F^{\mu\nu}u_{\nu}u_{\mu} = 0$$

as expected because

$u^{\mu}u_{\mu} = 0$.

Lorentz transformation of $\vec{E}$ and $\vec{B}$ fields

$$F'^{\mu\nu} = \Lambda^{\mu}{}_{\delta} \Lambda^{\nu}{}_{\rho} F^{\delta\rho}$$

by the transformation of tensors.

This yields

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel}$$

$$\vec{B}'_{\parallel} = \vec{B}_{\parallel}$$

$$\vec{E}'_{\perp} = \gamma \left(\vec{E} + \vec{v} \times \vec{B}\right)_{\perp}$$

$$\vec{B}'_{\perp} = \gamma \left(\vec{B} - \frac{\vec{v} \times \vec{E}}{c^2}\right)_{\perp}$$

For motion in the x-direction

$$E'_x = E_x \qquad B'_x = B_x$$

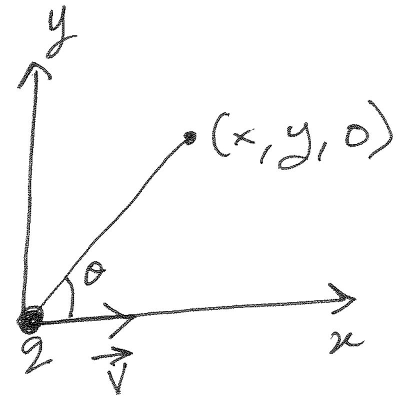
$$E'_y = \gamma \left(E_y - v B_z\right)$$

$$E'_z = \gamma \left(E_z + v B_y\right)$$

$$B'_y = \gamma \left(B_y + \frac{v E_z}{c^2}\right)$$

$$B'_z = \gamma \left(B_z - \frac{v E_y}{c^2}\right).$$

Field of a charge moving at a constant velocity



First examine the field in the x-y plane.

$$\vec{E}' = \frac{q\,\vec{r}'}{4\pi\epsilon_0 (r')^3}$$ in the frame of reference of the moving charge.

so $$E'_x = \frac{q\,x'}{4\pi\epsilon_0 \left(x'^2 + y'^2\right)^{3/2}}$$

$$E'_y = \frac{q\,y'}{4\pi\epsilon_0 \left(x'^2 + y'^2\right)^{3/2}}$$

We transform back to the lab frame by

$$x' = \gamma(x + vt) = \gamma x \qquad (t = 0)$$

$$y' = y$$

$$E_x = E'_x$$

$$E_y = \gamma\left(E'_y + vB'_z\right) = \gamma E'_y$$

$$B_x = B'_x = 0$$

$$B_y = \gamma\left(B'_y - \frac{v}{c^2}E'_z\right) = 0$$

$$B_z = \gamma\left(B'_z + \frac{v}{c^2}E'_y\right) = \frac{v}{c^2}E_y$$

Collecting everything

$$E_x = \frac{\gamma q x}{\left(\gamma^2 x^2 + y^2\right)^{3/2} 4\pi\epsilon_0}$$

$$E_y = \frac{\gamma q y}{\left(\gamma^2 x^2 + y^2\right)^{3/2} 4\pi\epsilon_0}.$$

$$\vec{B} = \frac{\vec{v}}{c^2} \times \vec{E}$$

Note: $\dfrac{E_y}{E_x} = \dfrac{y}{x} \rightarrow$ electric field is radial

~~Magnitude~~ Magnitude of electric field

$$E_0^2 = E_x^2 + E_y^2$$

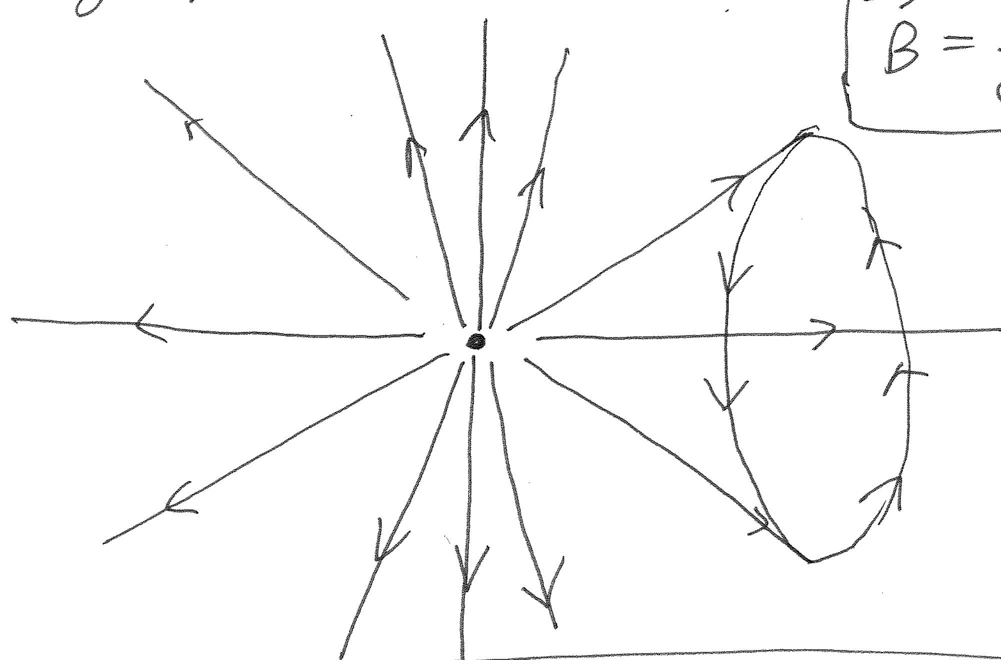
$$= \frac{q^2 (x^2+y^2)}{\gamma^4 \left(x^2+y^2-\frac{v^2}{c^2}y^2\right)^3} \frac{1}{(4\pi\epsilon_0)^2}$$

which yields

$$E_0 = \frac{q}{4\pi\epsilon_0 r^2} \frac{1}{\gamma^2\left(1-\frac{v^2}{c^2}\sin^2\theta\right)^{3/2}}.$$

We can now obtain the field at all x, y, z, by appealing to azimuthal symmetry

$$\boxed{\vec{B} = \frac{\vec{v}}{c^2} \times \vec{E}}$$



$$\boxed{\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{\hat{r}}{r^2} \frac{1-v^2/c^2}{\left(1-\frac{v^2}{c^2}\sin^2\theta\right)^{3/2}}}$$