# Action Principle

Advantages → maintain gauge ~~[illegible]~~ invariance

maintain relativistic invariance

quantize

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non-Abelian gauge fields

strings.

Non-relativistic particle in an electrostatic potential.

$$S[\vec{x}(t)] = \int_{t_1}^{t_2} dt \, L\left(\vec{x}(t), \vec{v} = \frac{d\vec{x}}{dt}\right)$$

Minimize $S$ over all $\vec{x}(t)$ obeying boundary conditions

Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i}$$

~~[illegible]~~ With $L = \frac{1}{2} m \left(\frac{dx_i}{dt}\right)^2 - q\,\Phi(x_i)$

we obtain

$$m \frac{d^2 x_i}{dt^2} = -q\,\partial_i \Phi$$ ✓

Relativistic generalisation

Note $\int dt\, \Phi(x_i) = \int dt\, d^3\!r\, \rho(r)\, \Phi(r)$

where $\rho(r) = q\, \delta^3(\vec{r} - \vec{x}(t))$.

The relativistic generalisation of this is clearly

$$\int dt\, d^3\!r\, A^\mu J_\mu$$

$$= \int dt \left[ \Phi(\vec{x}(t)) - \frac{d\vec{x}}{dt} \cdot \vec{A}(\vec{x}(t)) \right]$$

where recall

$$\vec{J}(r) = q \frac{d\vec{x}}{dt} \delta^3(\vec{r} - \vec{x}(t)).$$

For a relativistic particle in a EM field $A^\mu$ we propose the action

$$S[\vec{x}(t)] = -mc^2 \int d\tau - q \int d^4x\, A^\mu(x) J_\mu(x)$$

Reasons (i) $\int d\tau = \int dt \frac{d\tau}{dt} = \int dt \sqrt{1 - (\vec{v}/c)^2}$

$$\approx \int dt \left(1 - \frac{1}{2}\left(\frac{\vec{v}}{c}\right)^2\right) + \cdots$$

So the first term is constant $+ \frac{1}{2} \int dt\, m\vec{v}^{\,2} + \cdots$

✓ OK.

(ii) Both terms are Lorentz invariant

(iii) The second term is gauge-invariant

$$\int d^4x \, A^\mu J_\mu \longrightarrow \int d^4x \, (A^\mu + \partial^\mu \chi) J_\mu$$

$$= \int d^4x \, A^\mu J_\mu$$

$$- \int d^4x \, \chi \underbrace{\partial^\mu J_\mu}_{=0}$$

by current conservation.

Now rewrite the action in a more convenient manner

$$\int d^4x \, A^\mu J_\mu = \int dt \left( \Phi(\vec{x}(t)) - \frac{d\vec{x}}{dt} \cdot \vec{A}(\vec{x}(t)) \right)$$

$$= \int d\tau \left( \frac{dt}{d\tau} \Phi(\vec{x}(t)) - \frac{d\vec{x}}{d\tau} \cdot \vec{A}(\vec{x}(t)) \right)$$

$$= \int d\tau \, \frac{dx^\mu}{d\tau} A_\mu$$

So it is useful to think of the action as a functional of the 4 variables $x^\mu(\tau)$ rather than the 3 variables $\vec{x}(t)$.

To begin, we choose an <u>arbitrary</u> parametrization of the particle world line $x^\mu(\sigma)$.

Then we don't have to impose the constraint

$$\frac{dx^\mu}{d\sigma}\frac{dx_\mu}{d\sigma} \neq c^2$$

Also $d\tau^2 = \left(\frac{dx^\mu}{d\sigma}\frac{dx_\mu}{d\sigma}\right)\frac{d\sigma^2}{c^2}$

So the action is

$$S\left[x^\mu(\sigma)\right] = -mc\int d\sigma \left(\frac{dx^\mu}{d\sigma}\frac{dx_\mu}{d\sigma}\right)^{1/2}$$

$$- q\int d\sigma\, A_\mu(x)\,\frac{dx^\mu}{d\sigma}.$$

This action has a time reparametrization symmetry $\sigma \to f(\sigma) = \sigma'$

(check!).

Equation of motion

$$\cancel{\frac{\delta S}{\delta x^\mu(\sigma)} = mc^2 \frac{d}{d\sigma}\left(\frac{1}{2}\left(\frac{dx^\mu}{d\sigma}\frac{dx_\mu}{d\sigma}\right)^{-1/2}\right) + q\frac{d}{d\sigma}\left(A_\mu(x)\frac{dx^\mu}{d\sigma}\right)}$$

$$\frac{d}{d\sigma}\left(\frac{\delta S}{\delta \dot{x}_\mu}\right) = \frac{\delta S}{\delta x_\mu}$$

$$\Rightarrow \frac{d}{d\sigma}\left[-mc\left(\frac{dx_\nu}{d\sigma}\frac{dx^\nu}{d\sigma}\right)^{-1/2}\frac{dx^\mu}{d\sigma} - qA^\mu(x)\right] = \frac{\partial A_\nu}{\partial x_\mu}\frac{dx^\nu}{d\sigma}(-q)$$

Now set $\sigma = \tau$ the proper time

Then $\dfrac{dx_\nu}{d\tau}\dfrac{dx^\nu}{d\tau} = c^2$

<u>EOM</u>

$$\frac{d}{d\tau}\left[m\frac{dx^\mu}{d\tau} + qA^\mu\right] = q\frac{\partial A_\nu}{\partial x_\mu}\frac{dx^\nu}{d\tau}$$

$$m\frac{d^2x^\mu}{d\tau^2} + q\frac{\partial A^\mu}{\partial x^\nu}\frac{dx^\nu}{d\tau} = q\frac{\partial A_\nu}{\partial x_\mu}\frac{dx^\nu}{d\tau}$$

$$\Rightarrow \boxed{m\frac{du^\mu}{d\tau} = F^{\mu\nu}u_\nu}$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

$$u^\mu = \frac{dx^\mu}{d\tau}$$

# Action for the EM field

$$S_{EM} = -\frac{1}{4\mu_0}\int d^4x\, F^{\mu\nu}F_{\mu\nu}$$

$$= \int d^4x \left[ \frac{\epsilon_0}{2}\vec{E}^2 - \frac{1}{2\mu_0}\vec{B}^2 \right].$$

$$= \int d^4x \left[ \frac{\epsilon_0}{2}\left(\vec{\nabla}\Phi + \frac{\partial \vec{A}}{\partial t}\right)^2 - \frac{1}{2\mu_0}\left(\vec{\nabla}\times\vec{A}\right)^2 \right].$$

So action in the presence of a current $J^\mu$

$$S = \int d^4x \left[ -\frac{1}{4\mu_0}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) - A^\mu J_\mu \right].$$

Euler-Lagrange equations of motion

$$\frac{\delta S}{\delta A_\mu(x)} = 0 = -\frac{1}{\mu_0}\partial_\nu\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) - J^\mu$$

which yields $\partial_\mu F^{\mu\nu} = \mu_0 J$ ✓

The other Maxwell equation $\partial_\mu \epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} = 0$

just follows from $F_{\rho\sigma} = \partial_\rho A_\sigma - \partial_\sigma A_\rho$.

# Hamiltonian

From the Lagrangian $L(x_i, \dot{x}_i)$ we define momenta $p_i = \frac{\partial L}{\partial \dot{x}_i}$, and the Hamiltonian is $H(x_i, p_i) = p_i \dot{q}_i - L$.

The equations of motion are

$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x_i}$$

## For a relativistic particle in a EM field

$$L(\vec{x}, \dot{\vec{x}}) = -mc^2\sqrt{1-\frac{\dot{\vec{x}}^2}{c^2}} - q\Phi(x) + q\vec{A}(x)\cdot\frac{d\vec{x}}{dt}$$

So

$$\vec{p} = \frac{\partial L}{\partial \dot{\vec{x}}} = \frac{m\dot{\vec{x}}}{\sqrt{1-\frac{\dot{\vec{x}}^2}{c^2}}} + q\vec{A}$$

So (now using $\dot{\vec{x}} = \vec{v}$ and $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$)

$$H = \vec{p}\cdot\vec{v} - L = (\gamma m\vec{v} + q\vec{A})\cdot\vec{v} + \frac{mc^2}{\gamma} + q\Phi - q\vec{A}\cdot\vec{v}$$

$$= \gamma mc^2 + q\Phi$$

$$\boxed{H = c\sqrt{(\vec{p}-q\vec{A})^2 + m^2c^2} + q\Phi}$$

So the vector potential $\vec{A}$ appears as

$$\vec{p} \longrightarrow \vec{p} - q\vec{A}.$$

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See Section 24.5 of Zangwill for a general discussion. We will only ~~constrict~~ consider the Coulomb gauge

$$\boxed{\vec{\nabla}\cdot\vec{A} = 0}$$

Hamiltonian for the EM field

$$L_{EM} = \int d^3x \left[ \frac{\epsilon_0}{2}\left(\vec{\nabla}\Phi + \frac{\partial \vec{A}}{\partial t}\right)^2 - \frac{1}{2\mu_0}(\nabla \times A)^2 \right]$$

$$= \int d^3x \left[ \frac{\epsilon_0}{2}\left(\vec{\nabla}\Phi\right)^2 + \frac{\epsilon_0}{2}\left(\frac{\partial \vec{A}}{\partial t}\right)^2 - \frac{1}{2\mu}\left(\vec{\nabla}\times\vec{A}\right)^2 \right]$$

There is no dependence on $\frac{\partial \Phi}{\partial t} \Rightarrow \Phi$ is not a dynamical variable but tied to the charge density

$$\Phi(r,t) = \frac{1}{4\pi\epsilon_0}\int d^3r' \frac{\rho(r')}{|r - r'|}$$

We only have a Hamiltonian for $\vec{A}$ with the constraint $\vec{\nabla}\cdot\vec{A} = 0$.

'Momentum' conjugate to $\vec{A}$

$$\vec{\Pi} = \frac{\delta L_{EM}}{\delta \dot{\vec{A}}(t)} = \epsilon_0 \frac{\partial \vec{A}}{\partial t}$$

So

$$H_{EM}(\vec{A}) = \int d^3r \left[ \frac{1}{2\epsilon_0} \vec{\Pi}^2 + \frac{\epsilon_0}{2} (\vec{\nabla} \times \vec{A})^2 \right].$$

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Hamiltonian for everything : particles with charge $q_k$ and EM field

$$\boxed{H = \sum_k c \sqrt{(\vec{p}_k - q_k \vec{A}(\vec{r}_k))^2 + m^2 c^2} + \frac{1}{8\pi\epsilon_0} \int d^3r \, d^3r' \frac{\rho(r)\rho(r')}{|r - r'|} + \int d^3r \left[ \frac{1}{2\epsilon_0} \vec{\Pi}^2 + \frac{\epsilon_0}{2} (\nabla \times \vec{A})^2 \right]}$$

where $\vec{\Pi}(x,t)$ is canonically conjugate of $\vec{A}(x,t)$ and

$$\vec{\nabla} \cdot \vec{A} = 0$$