Gauge invariance and Quantum Mechanics

Schroedinger equation (Relativistic case similar)

$$\left[\frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla} - q\vec{A}\right)^2 + q\Phi\right]\psi = i\hbar\frac{\partial\psi}{\partial t}.$$

Gauge transformation

$$\vec{A} \longrightarrow \vec{A} + \vec{\nabla}\Lambda$$

$$\Phi \longrightarrow \Phi - \frac{\partial\Lambda}{\partial t}$$

$$\psi \longrightarrow \psi \exp\left(\frac{iq}{\hbar}\Lambda\right)$$

Verify that the Schroedinger equation remains invariant!

The wavefunction is <u>not gauge invariant</u>

Dirac Monopoles

Are there particles of magnetic charge g such that $\vec{B} = \frac{g\hat{r}}{4\pi r^2}$?

Then $\vec{\nabla}\cdot\vec{B} = \rho_m$ magnetic charge density.

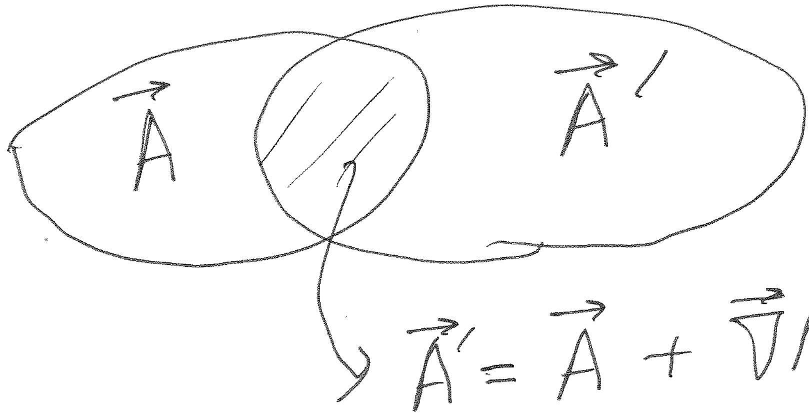
No contradiction in classical Maxwell's equation.

But we have learnt that the vector potential $\vec{A}$ is essential for quantum mechanics. → How do we define $\vec{A}$ near a magnetic monopole?

$$\vec{B} = \vec{\nabla}\times\vec{A} \Rightarrow \nabla\cdot B = 0 \; ??$$

$\Rightarrow \vec{A}$ is globally singular.

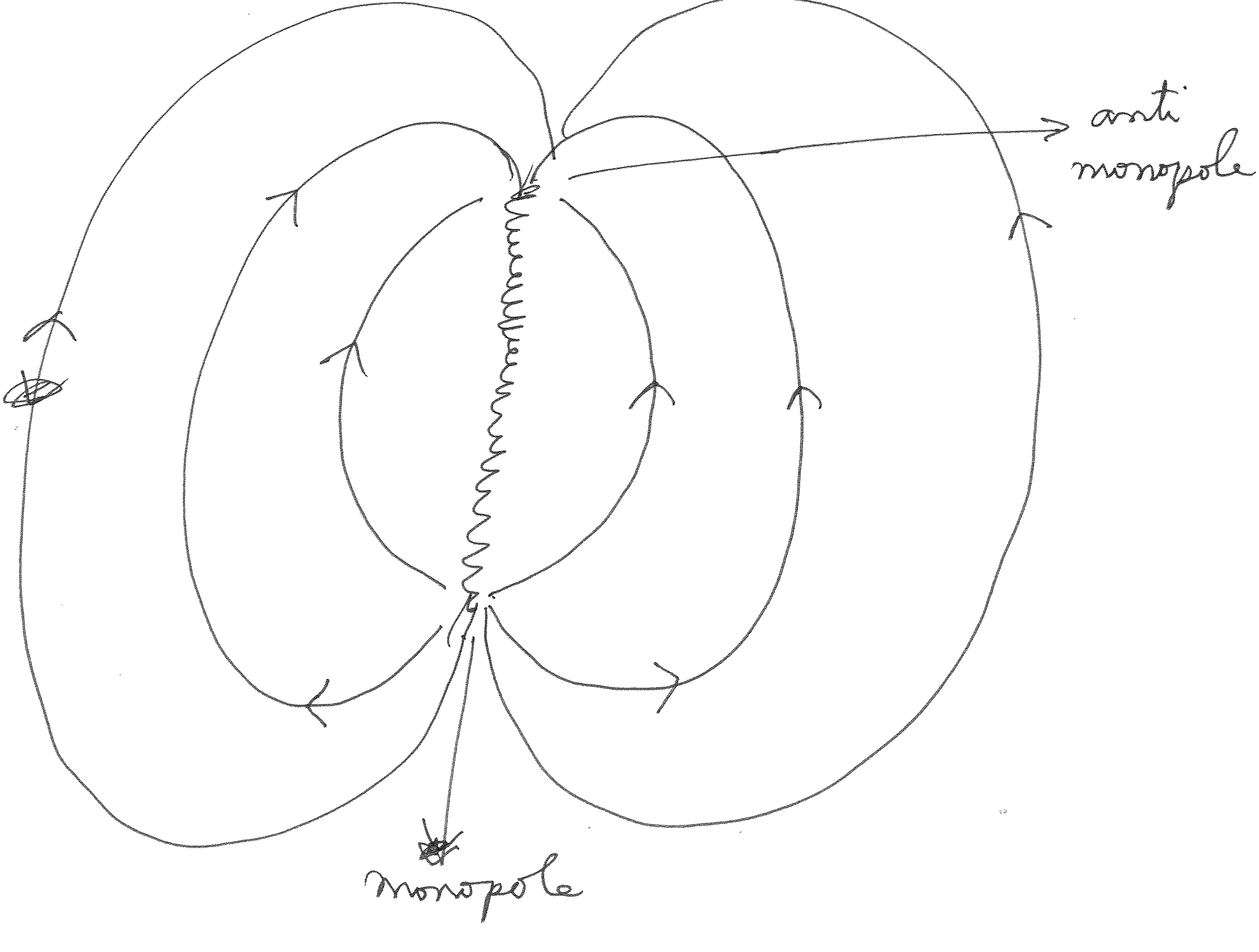
<u>But</u> $\vec{A}$ is well defined in patches and different patches are related by gauge transformations.



$$\vec{A}' = \vec{A} + \vec{\nabla}\Lambda .$$

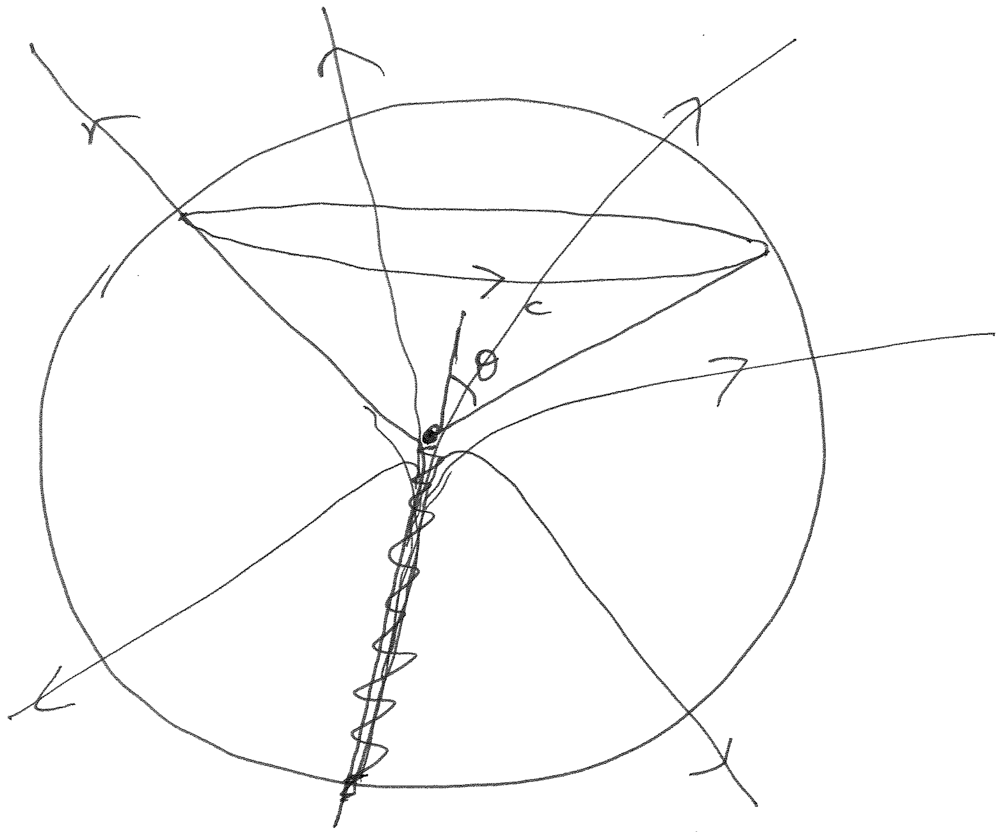
Can we do this globally around a magnetic monopole?

One way to create a monople – anti monopole pair → an infinitely thin solenoid



Vector potential around a monopole

$$A_\phi = \frac{g}{4\pi r} \frac{(1-\cos\theta)}{\sin\theta}$$



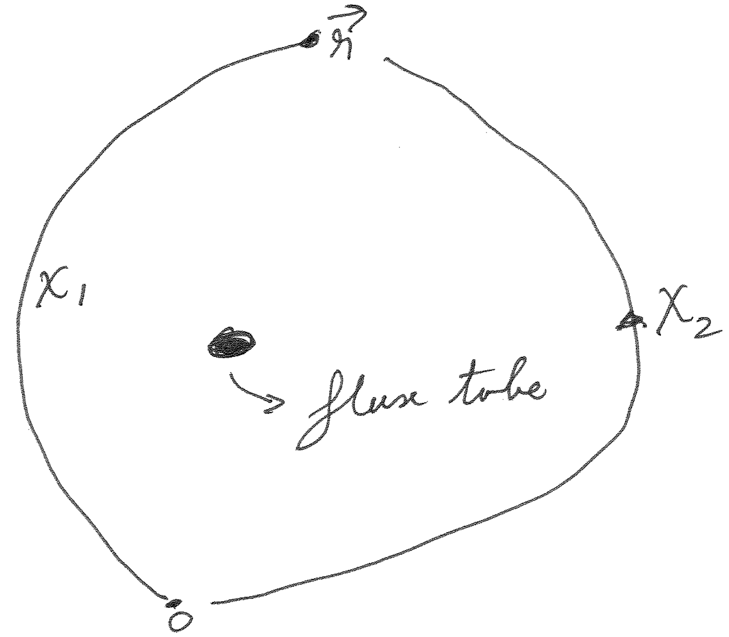
Then $\vec{\nabla} \times \vec{A} = \dfrac{g}{4\pi r^2} \hat{r}$ ✓

Also let us evaluate

$$\oint_C \vec{A}\cdot\vec{dl} = \int d\phi \, r \sin\theta \, A_\phi = \frac{g}{2}(1-\cos\theta)$$

$\Rightarrow$ There is a ~~Dir~~ Dirac string carrying magnetic flux g towards the south pole

Dirac : It is possible to make the flux invisible



Outside flux tube $\vec{B} = \vec{\nabla} \times \vec{A} = 0$.

$$\Rightarrow \vec{A} = \vec{\nabla} \chi$$

Locally we have $\chi = \int_0^{\vec{r}} d\vec{r} \cdot \vec{A}$

Knowing χ, the wavefunction remains unchanged apart from $\psi \to \psi \exp\left(-\frac{iq}{\hbar}\chi\right)$.

But depending upon the path the phase factor is $\exp\left(-\frac{iq}{\hbar}\chi_1\right)$ or $\exp\left(-\frac{iq}{\hbar}\chi_2\right)$.

Flux is invisible provided

$$\frac{q}{\hbar}(\chi_1 - \chi_2) = 2\pi n$$

where n is an integer

But $\chi_1 - \chi_2 = \oint \vec{A} \cdot dl = \text{Flux} = g$

So $\boxed{g q = 2\pi \hbar n}$

- Dirac quantization condition.
- The existence of one monopole in the universe is sufficient to explain charge quantization
- Grand unified theories (and string theory) always have monopoles.