Maxwell's Equations in Macroscopic Media

$\vec{e}(r,t) \to$ microscopic electric field

$\vec{b}(r,t) \to$ " magnetic field

$\eta(r,t) \to$ microscopic charge density

$j(r,t) \to$ " current density

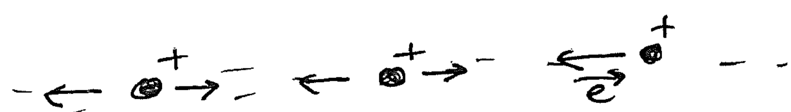
$$\boxed{\begin{array}{l} \vec{b} = \nabla\times a \\ \vec{e} = -\nabla\phi - \dfrac{\partial a}{\partial t} \end{array}}$$

$$\nabla\cdot e = \eta/\epsilon_0 \qquad \nabla\cdot b = 0$$

$$\nabla\times e = -\frac{\partial b}{\partial t} \qquad \nabla\times b = \mu_0 j + \frac{1}{c^2}\frac{\partial e}{\partial t}.$$

There are strong spatial oscillations in $\vec{e}$ in any metal ~~of~~ or insulator

insulator



We are only interested in spatial averages

e.g. $$\vec{E}(r,t) = \left\langle \vec{e}(r,t)\right\rangle_{R_0}$$

$$= \frac{1}{(2\pi R_0^2)^{3/2}} \int d^3r'\, \vec{e}(r',t)\, e^{-|r'-\vec{r}|^2/2R_0^2}$$

Then we can show

e.g. $$\left\langle \vec{\nabla}\cdot\vec{e}\right\rangle_{R_0} = \vec{\nabla}\cdot\vec{E}$$

$$\vec{\nabla}\cdot\vec{E} \propto \int d^3r'\, \vec{e}(r')\, \underbrace{\nabla_r}_{=-\nabla_{r'}} e^{-|r-r'|^2/2R_0^2}$$

and integrate by parts

$$= \langle \nabla\cdot e \rangle$$

In this manner

$$\vec{\nabla}\cdot\vec{E} = \langle \eta \rangle/\epsilon_0 \,, \quad \nabla\cdot B = 0$$

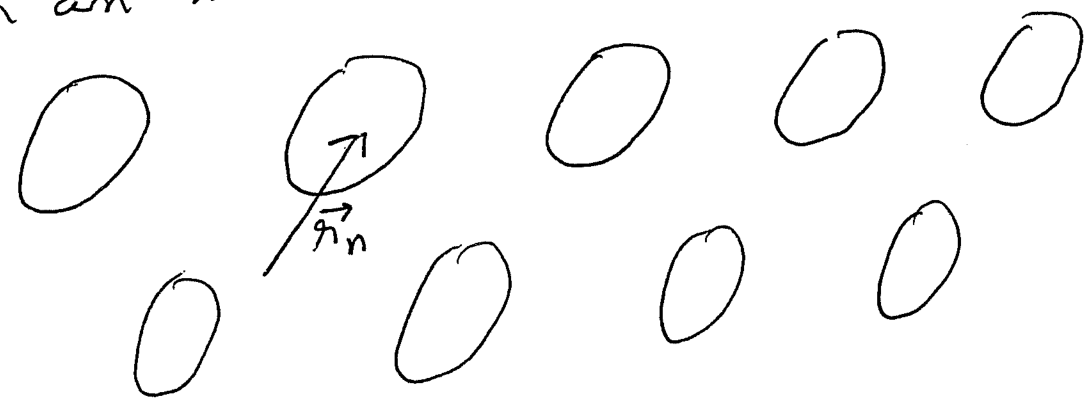
$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \,, \quad \nabla\times B = \mu_0\langle j\rangle + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}.$$

We assume η and $\vec{j}$ are divided into "free charges" and "bound charges".

We will characterize the bound charges in a simple "molecular" model.

~~and ignore free~~

In an insulator, there are no free charges.



Each molecule at position $\vec{r}_n$ is characterised by a

<u>charge</u> $\quad q_n = \int d^3r' \, \eta(r')$

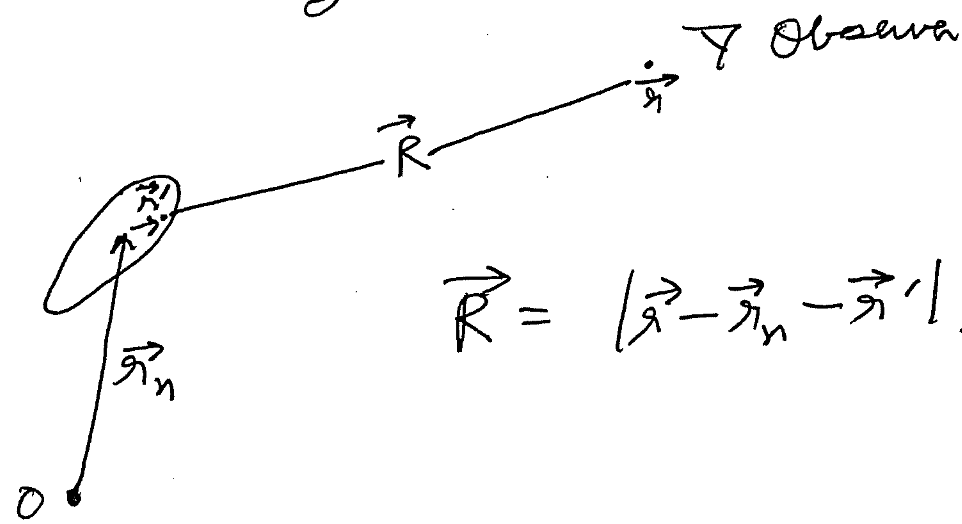
<u>electric dipole moment</u>

$$\vec{p}_n = \int d^3r' \, \vec{r}' \, \eta(\vec{r}')$$

and a <u>magnetic dipole moment</u>

$$\vec{m}_n = \frac{1}{2} \int d^3r' \, \vec{r}' \times \vec{J}(\vec{r}').$$

Note, actually $q_n(t)$, $\vec{p}_n(t)$, $\vec{m}_n(t)$ are functions of time determined by the instantaneous values of $\vec{\eta}(r', t)$ and $\vec{J}(r', t)$.



$$\vec{R} = |\vec{r} - \vec{r}_n - \vec{r}'|.$$

In the Lorentz gauge, electrostatic potential due to molecule at $\vec{r}$ is

$$\phi(\vec{r}, t) = \int d^3r' \, \frac{\eta(\vec{r}', t - R/c)}{4\pi\varepsilon_0 R}$$ retarded time

We neglect retardation effects <u>within</u> a single molecule, and so $t - R/c \approx t - \frac{|\vec{r}-\vec{r}_n|}{c}$.

Then, using earlier analysis we have

$$\boxed{\phi(\vec{r}, t) = \frac{1}{|\vec{r}-\vec{r}_n|} \times \frac{q_n\left(t - \frac{|\vec{r}-\vec{r}_n|}{c}\right)}{4\pi\epsilon_0} + \frac{\vec{p}_n\left(t - \frac{|\vec{r}-\vec{r}_n|}{c}\right) \cdot (\vec{r}-\vec{r}_n)}{4\pi\epsilon_0 |\vec{r}-\vec{r}_n|^3}}$$

and

$$\vec{a}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int d^3r' \, \vec{J}\left(r', t - \frac{|r-\vec{r}_n|}{c}\right)$$

$$+ \frac{\mu_0 (\vec{r}-\vec{r}_n) \cdot}{4\pi |\vec{r}-\vec{r}_n|^3} \int d^3r' \, \vec{r}' \vec{J}\left(\vec{r}', t - \frac{|r-r_n|}{c}\right)$$

Type something

All expressions ... t_{ret} ... $\frac{\quad}{c}$

and we will leave this implicit.

Using $\quad \frac{\partial \rho}{\partial t} + \partial_i j_i = 0.$

$$\partial_j (r_k j_j) = \delta_{jk} j_j + r_k \partial_j j_j$$

$$= j_k - r_k \frac{\partial \rho}{\partial t}.$$

So $\quad \int d^3 r' \vec{J} = \int d^3 r' \, \vec{r}' \frac{\partial \rho}{\partial t} = \underline{\underline{\frac{\partial \vec{p}_n}{\partial t}}}.$

Similarly as before

Text

$$\int d^3 r' \, r'_j \, j_l(r') = \epsilon_{jln} \, m_n + \cancel{\frac{\partial}{\partial t}(\text{quadrupole moment})} \quad \text{ignore.}$$

In this manner we obtain

$$\boxed{\begin{aligned} \vec{a}(\vec{r}, t) = {} & \frac{\mu_0}{4\pi |\vec{r} - \vec{r}_n|} \frac{\partial}{\partial t} \vec{p}_n\left(t - \frac{|r_n - r|}{c}\right) \\ & - \frac{\mu_0}{4\pi} \frac{(\vec{r} - \vec{r}_n) \times \vec{m}_n\left(t - \frac{|r - r_n|}{c}\right)}{|\vec{r} - r_n|^3} \end{aligned}}$$

Now we

(i) sum over all molecules at positions $\vec{r}_n$

(ii) introduce ρ_{mol} → molecular charge density

$$\rho_{mol}(\vec{r}) = \sum_n q_n \delta(\vec{r}-\vec{r}_n)$$

$$\vec{P}_{mol}(\vec{r}) = \sum_n \vec{p}_n \delta(\vec{r}-\vec{r}_n)$$

molecular electric dipole density

$$\vec{M}_{mol}(\vec{r}) = \sum_n \vec{m}_n \delta(\vec{r}-\vec{r}_n)$$

molecular magnetic dipole density

(iii) Use $\frac{\vec{r}-\vec{r}_n}{|\vec{r}_n-\vec{r}|^3} = \nabla_n \frac{1}{|\vec{r}-\vec{r}_n|}$

(iv) Write $\sum_n p_n f(r_n)$

$$= \int d^3r' \, \vec{P}_{mol}(r') \, f(r')$$

(v) Integrate by parts

Then we obtain

$$\phi(\vec{r},t) = \int d^3r' \frac{\left[\rho_{mol}(r') + \rho_{ext}(r') - \vec{\nabla}'\cdot\vec{P}_{mol}(r')\right]_{ret}}{4\pi\epsilon_0 |r - r'|}$$

and

$$\vec{a}(\vec{r},t) = \frac{\mu_0}{4\pi}\int d^3\vec{r}' \frac{\left[\frac{\partial \vec{P}_{mol}(\vec{r}')}{\partial t} + \vec{J}_{ext}(\vec{r}') + \vec{\nabla}'\times\vec{M}_{mol}\right]_{ret}}{|\vec{r} - \vec{r}'|}$$

Finally we average over ~~re~~ regions of size R_0.

$$\langle \vec{P}_{mol} \rangle_{R_0} = \vec{P}, \text{ the polarization}$$

$$\langle \vec{M}_{mol} \rangle_{R_0} = \vec{M}, \text{ the magnetization}$$

$$\langle \rho_{mol}(r') \rangle_{R_0} + \rho_{ext}(r') = \rho_{free}$$

Then undoing the derivation of the expressions for ϕ and $\vec{a}$ we obtain

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\left(\rho_{free} - \vec{\nabla}\cdot\vec{P}\right)$$

$$\vec{\nabla}\times\vec{B} = \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} + \mu_0\left(\vec{J}_{ext} + \frac{\partial\vec{P}}{\partial t} + \vec{\nabla}\times\vec{M}\right).$$

and the source free ~~m~~ Maxwell's equations remain the same.

We define

$$\boxed{\begin{aligned} \vec{D} &= \epsilon_0 \vec{E} + \vec{P} \\ \vec{H} &= \frac{1}{\mu_0} \vec{B} - \vec{M} \end{aligned}}$$

"displacement field"

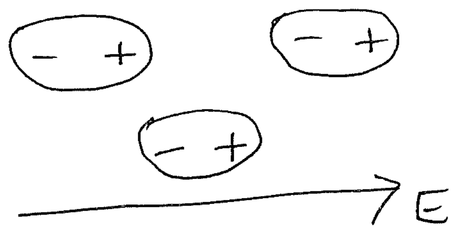
$\vec{H} \rightarrow$ "magnetic field"

$\vec{B} \rightarrow$ "magnetic induction"

Then

$$\boxed{\begin{aligned} \vec{\nabla} \cdot \vec{D} &= \rho_{free} \\ \vec{\nabla} \times \vec{H} &= \vec{J}_{ext} + \frac{\partial \vec{D}}{\partial t} \\ \vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \end{aligned}}$$

In most insulators $\vec{P}$ is produced by an applied electric field



As the response is weak, we write

$$\vec{P} = \epsilon_0 \chi \vec{E}$$

$\chi \rightarrow$ electric susceptibility > 0

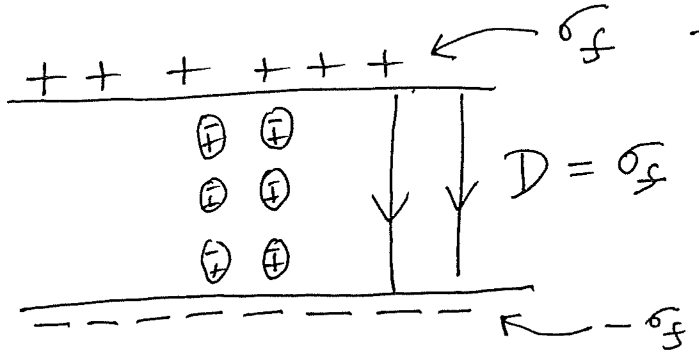
$$\vec{D} = \epsilon_0 \vec{E} + P = \epsilon \vec{E}$$

with $\epsilon = \epsilon_0 (1+\chi)$

$\epsilon \longrightarrow$ permittivity

$K = \dfrac{\epsilon}{\epsilon_0} = 1 + \chi$ is the <u>dielectric constant.</u>

<u>$K > 1$</u>.

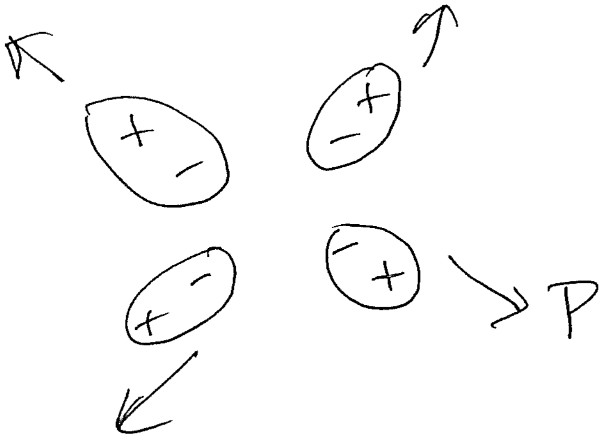


$$D = \sigma_f \;;\quad E = \frac{\sigma_f}{\epsilon}$$

Net charge on the top surface $= \epsilon_0 E = \sigma_f \epsilon_0/\epsilon$

Net charge = Free charge + polarization charge

$\Rightarrow$ Polarization charge $= \sigma_f \dfrac{\epsilon_0}{\epsilon} - \sigma_f = -\sigma_f \dfrac{\chi}{1+\chi} = \sigma_f \dfrac{(1-K)}{K}$.

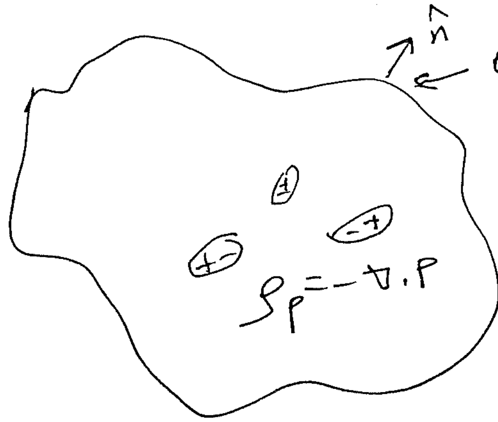


$$\rho_P = -\vec{\nabla}\cdot\vec{P}$$

~~σ_p = surface polarization density~~

Similarly, on a surface, surface polarization charge density $\sigma_p = \vec{P}\cdot\vec{n}$ where $\vec{n}$ is the normal <u>away</u> from the sample.

$\hat{n}$ $\sigma_p = \hat{n}\cdot\vec{P}$



In any insulating sample

$$\int d^3r\, \rho_P + \int dS\, \sigma_p = 0.$$

Follows from Gauss's law and definitions above.

For magnetic fields we have

$$\vec{M} = \chi_m \vec{H}$$ where $\chi_m \rightarrow$ magnetic susceptibility.

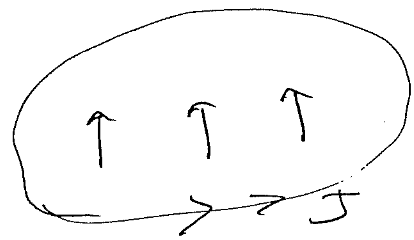
So $\vec{B} = \mu \vec{H}$

$\mu = \mu_0 (1 + \chi_m)$ is the <u>permeability</u>.

and $\vec{M} = \dfrac{1}{\mu_0} \dfrac{\chi_m}{1+\chi_m} \vec{B}$

<u>Paramagnetism</u> $\chi_m > 0$

Usually due to <u>spins</u>

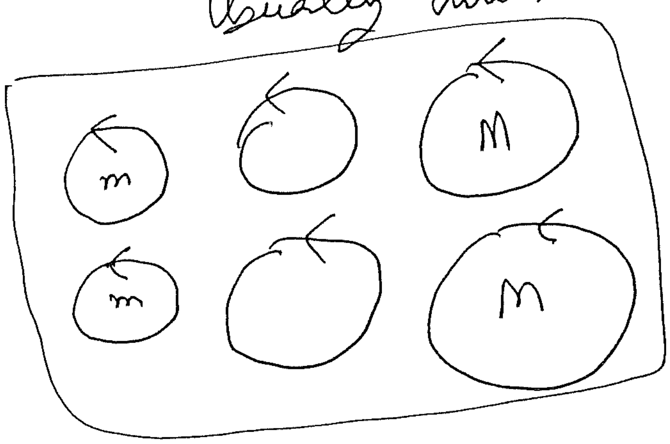


<u>Diamagnetism</u> ~~$\chi_m < 0$~~ $-1 < \chi_m < 0$

$\chi_m \rightarrow -1$, perfect diamagnet with $\vec{B} = 0$

and $\vec{M} = -\vec{H} \Rightarrow$ a superconductor!

Usually due to orbital currents



$$\vec{J}_{bound} = \vec{\nabla} \times \vec{M}$$

Surface current

$$\vec{K}_{bound} = \vec{M} \times \vec{n}$$

Maxwell's Equations

$$\vec{\nabla}\cdot\vec{D} = \rho_{free} \qquad \vec{\nabla}\times\vec{H} = \vec{J}_{free} + \frac{\partial\vec{D}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{D} = \epsilon\vec{E}$$

$$\vec{B} = \mu\vec{H}$$