Phys 232 Problem Set 1

Released: 1/30/2020
Due: 2/10/2020

Problem 1

(a) Prove that

$$\delta(ax) = \frac{1}{|a|}\delta(x),\ a \neq 0 \tag{1}$$

(b) Use the identity in part (a) to prove that

$$\delta(g(x)) = \sum_m \frac{1}{|g'(x_m)|}\delta(x - x_m), \quad \text{where}\, g(x_m) = 0\, \text{and}\, g'(x_m) \neq 0 \tag{2}$$

(c) Show that

$$\int_{-\infty}^{\infty} dx' f(x')\delta'(x' - x) = -f'(x) \tag{3}$$

Problem 2

Consider a collection of N point particles fixed in space, each with time varying charge $q_i(t)$. The charge density can be expressed as

$$\rho(\boldsymbol{r}, t) = \sum_{i=1}^{N} q_i(t)\delta(\boldsymbol{r} - \boldsymbol{r}_i) \tag{4}$$

Suppose that $\boldsymbol{E}(\boldsymbol{r}, t = 0) = \boldsymbol{B}(\boldsymbol{r}, t = 0) = 0$ and

$$\boldsymbol{E}(\boldsymbol{r}, t) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N} q_i(t)\frac{\boldsymbol{r} - \boldsymbol{r}_i}{|\boldsymbol{r} - \boldsymbol{r}_i|^3} \tag{5}$$

(a) Show that the current density

$$\boldsymbol{J}(\boldsymbol{r}, t) = -\sum_{i}^{N} \frac{dq_i(t)}{dt}\frac{1}{4\pi}\frac{\boldsymbol{r} - \boldsymbol{r}_i}{|\boldsymbol{r} - \boldsymbol{r}_i|^3} \tag{6}$$

satisfies the continuity equation.

(b) Find $\boldsymbol{B}(\boldsymbol{r}, t)$ and show that this field and $\boldsymbol{E}(\boldsymbol{r}, t)$ satisfy all four Maxwell equations

Problem 3

If the photon had a mass m, the electric field would remain $\boldsymbol{E} = -\nabla\varphi$ but Poisson's equation would change to include a length $L = \hbar/mc$ i.e.

$$\nabla^2\varphi = -\frac{\rho}{\epsilon_0} + \frac{\varphi}{L^2} \tag{7}$$

Experimental searches for m use a geometry first employed by Cavendish where two concentric conducting shells (radii $r_1 < r_2$) are maintained at a common potential Φ by an infinitesimally thin connecting wire. When $m = 0$, all excess charge resides on the outside of the outer shell; no charge accumulates on the inner shell.

(a) Use the substitution $\varphi(r) = u(r)/r$ to solve the generalized Poisson equation above in the space between the shells. Also, find the electric field in this region.

(b) Use the generalization of Gauss' law implied by the modified Poisson equation to find the charge Q on the inner shell.

(c) Show that, to leading order when $L \to \infty$,

$$Q \approx \frac{2\pi\epsilon_0}{3}\frac{r_1\Phi}{L^2}\left(\frac{r_2}{L}\right)^2\left(1+\frac{r_1}{r_2}\right) \tag{8}$$

Problem 4

Second derivatives are difficult to calculate numerically with high accuracy. Therefore, if both the fields and the potentials (Lorenz gauge) are of interest, a convenient equation to integrate is

$$\frac{\partial \boldsymbol{A}}{\partial t} = -\boldsymbol{E} - \nabla\varphi \tag{9}$$

(a) Let $\mathcal{C}(\boldsymbol{r}, t) = \nabla \cdot \boldsymbol{E} - \frac{\rho}{\epsilon}$ and let the initial conditions satisfy $\mathcal{C}(\boldsymbol{r}, t = 0) = 0$. If this Gauss' law condition is maintained, show that the equation above combined with the two equations below produces fields that satisfy all four Maxwell equations and properly defined potentials:

$$\frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} = \nabla \times (\nabla \times \boldsymbol{A}) - \mu_0 \boldsymbol{J} \quad \text{and} \quad \frac{\partial \varphi}{\partial t} = -c^2 \nabla \cdot \boldsymbol{A} \tag{10}$$

(b) Show that the three equations above imply that $\frac{\partial \mathcal{C}}{\partial t} = 0$. Hence, any initial differences from zero (due to numerical noise) are frozen onto the computational grid (which is not a good thing).

(c) Show that the two equations in (a) can be replaced by $\dot{\varphi} = -c^2\Gamma$ with

$$\frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} = -\nabla^2 \boldsymbol{A} + \nabla\Gamma - \mu_0 \boldsymbol{J} \quad \text{and} \quad \frac{\partial \Gamma}{\partial t} = -\frac{\rho}{\epsilon_0} - \nabla^2\varphi \tag{11}$$

(d) Show that $\dot{\boldsymbol{A}} = -\boldsymbol{E} - \nabla\varphi$ and the three equations in part (c) imply that $\frac{\partial^2 \mathcal{C}}{\partial t^2} = c^2 \nabla^2 \mathcal{C}$. Hence, any initial differences from zero propagate out of the computational grid at the speed of light. For this reason, set (c) is preferred to set (a) for numerical work.

Problem 5

(a) Confirm that $\varphi(\boldsymbol{r}) = -\boldsymbol{r} \cdot \boldsymbol{E}$ and $\boldsymbol{A} = -\frac{1}{2}\boldsymbol{r} \times \boldsymbol{B}$ are acceptable scalar and vector potentials, respectively for a constant electric field $\boldsymbol{E}$ and a constant magnetic field $\boldsymbol{B}$.

(b) By direct computation of $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ and $\boldsymbol{E} = -\nabla\varphi - \frac{\partial \boldsymbol{A}}{\partial t}$, prove that the generalizations of the formulae in part (a) to arbitrary time-dependent fields are

$$\varphi(\boldsymbol{r}, t) = -\boldsymbol{r} \cdot \int_0^1 d\lambda E(\lambda \boldsymbol{r}, t) \quad \text{and} \quad \boldsymbol{A}(\boldsymbol{r}, t) = -\int_0^1 d\lambda (\lambda \boldsymbol{r} \times \boldsymbol{B}(\lambda \boldsymbol{r}, t)) \tag{12}$$

Hint: first prove that

$$\frac{d}{d\lambda}\boldsymbol{G}(\lambda r) = \frac{1}{\lambda}(\boldsymbol{r} \cdot \nabla)\boldsymbol{G}(\lambda \boldsymbol{r}) \tag{13}$$

for any vector field $\boldsymbol{G}$

Problem 6

An early competitor of the Big Bang theory postulates the "continuous creation" of charged matter at a (very small) constant rate R at every point in space. In such a theory, the continuity equation is replaced by

$$\nabla \cdot \boldsymbol{J} + \frac{\partial \rho}{\partial t} = R \tag{14}$$

(a) For this to be true, it is necessary to alter the source terms in the Maxwell equations. Show that it is sufficient to modify Gauss' law to

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} - \lambda \varphi \tag{15}$$

and the Ampere-Maxwell law to

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \frac{1}{c^2} \frac{\partial \boldsymbol{E}}{\partial t} - \lambda \boldsymbol{A} \tag{16}$$

Here, λ is a constant and φ and $\boldsymbol{A}$ are the usual scalar and vector potentials. Is this theory gauge invariant?

(b) Confirm that a spherically symmetric solution of the new equations exists with

$$\boldsymbol{A}(r,t) = \boldsymbol{r} f(r,t) \text{ and } \varphi(r,t) = \varphi_0 \tag{17}$$

where $f(r,t)$ is a scale function and φ_0 is a constant.

(c) Show that the only non-singular solution to the partial differential equation satisfied by $f(r,t)$ is a constant.

(d) Show that the velocity of the charge created by this theory, $v = \boldsymbol{J}/\rho$, is a linear function of r. This agrees with Hubble's famous observations.