Phys 232 Problem Set 2

Released: 2/12/2020
Due: 2/24/2020

Problem 1

An example of the preservation of causality and finite speed of propagation in spite of the use of Coulomb gauge is afforded by dipole source that is flashed on and off at $t = 0$. The effective charge and current densities are

$$\rho(\boldsymbol{r}, t) = \delta(x)\delta(y)\delta'(z)\delta(t) \tag{1}$$
$$J(\boldsymbol{r}, t) = -\hat{\boldsymbol{z}}\delta(x)\delta(y)\delta(z)\delta'(t) \tag{2}$$

This dipole is of unit strength and it points in the negative z direction

(a) Show that the instantaneous Coulomb potential is

$$\Phi(\boldsymbol{r}, t) = -\frac{1}{4\pi\epsilon_0}\delta(t)\frac{z}{r^3} \tag{3}$$

(b) Show that the transverse current J_t is

$$J_t(\boldsymbol{r}, t) = -\delta'(t)\left[\frac{2}{3}\hat{\boldsymbol{z}}\delta(\boldsymbol{r}) - \frac{\hat{\boldsymbol{z}}}{4\pi r^3} + \frac{3}{4\pi r^3}\hat{\boldsymbol{r}}(\hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{r}})\right] \tag{4}$$

where the factor of $\frac{2}{3}$ comes from treating the gradient of z/r^3 according to

$$E(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{3\hat{\boldsymbol{r}}(\boldsymbol{p}\cdot\hat{\boldsymbol{r}}) - \boldsymbol{p}}{|\boldsymbol{r}-\boldsymbol{r}'|^3} - \frac{4\pi}{3}\boldsymbol{p}\delta(\boldsymbol{r}-\boldsymbol{r}')\right] \tag{5}$$

(c) Show that the electric and magnetic fields are causal and that the electric field is

$$\begin{aligned} E(\boldsymbol{r}, t) = {} & \frac{1}{4\pi\epsilon_0}\frac{c\sin\theta\cos\theta\cos\phi}{r}\left[-\delta''(r-ct) + \frac{3}{r^2}\delta(r-ct) - \frac{3}{r}\delta'(r-ct)\right]\hat{\boldsymbol{x}} \\ & + \frac{1}{4\pi\epsilon_0}\frac{c\sin\theta\cos\theta\sin\phi}{r}\left[-\delta''(r-ct) + \frac{3}{r^2}\delta(r-ct) - \frac{3}{r}\delta'(r-ct)\right]\hat{\boldsymbol{y}} \\ & + \frac{1}{4\pi\epsilon_0}\frac{cz}{r^2}\left[\sin\theta^2\delta''(r-ct) + (3\cos^2\theta - 1)\left(\delta'(r-ct) - \frac{\delta(r-ct)}{r^2}\right)\right]\hat{\boldsymbol{z}} \end{aligned} \tag{6}$$

Hint: You can use

$$\delta(x) = \lim_{\epsilon\to 0}\frac{1}{\pi}\frac{\epsilon}{x^2+\epsilon^2} \tag{7}$$

to explain why Eqn. 1 and Eqn. 2 describe a mathematical idealization of a dipole which is flashed on and off at $t = 0$. These approximations may also help you make sense of the fields derived in part (c).

Problem 2

An infinitely long straight wire on the z-axis has a circular cross section and obeys $\boldsymbol{J}(\omega) = \sigma_0 \boldsymbol{E}(\omega)$ for all $\rho \leq a$. After initial transients, the charge density $\rho(\boldsymbol{r}, t) \equiv 0$ and the current $I(t) = I_0 \cos \omega t$ everywhere inside the wire.

(a) Solve an appropriate Helmholtz equation and find the exact $\boldsymbol{E}(\boldsymbol{r}, t)$ inside the wire. Express the amplitude of the field in terms of I_0.

(b) Solve an appropriate Helmholtz equation and find $\boldsymbol{E}$ an $\boldsymbol{B}$ exactly outside the wire.

(c) Use Poynting's theorem to show that the normal component of the time-averaged Poynting vector $\langle \S \rangle$ evaluated on any cylindrical surface concentric with the wire always points toward the z-axis.

(d) Use the Poynting vector to calculate the rate at which energy is lost to ohmic heating per unit length of the wire.

(e) Use the Poynting vector to calculate the rate at which energy is lost to radiation per unit length of wire. How is this result consistent with conservation of energy and the answer to part (c)?

Problem 3

Two small antennas a distance L apart along the z axis oscillate in phase at the same angular frequency ω and amplitude α

$$\boldsymbol{J}(\boldsymbol{r}, t) = \alpha \hat{\boldsymbol{z}} \delta^3 \left(\boldsymbol{r} - \frac{L}{2} \hat{\boldsymbol{z}} \right) \cos \omega t + \alpha \hat{\boldsymbol{z}} \delta^3 \left(\boldsymbol{r} + \frac{L}{2} \hat{\boldsymbol{z}} \right) \cos \omega t \tag{8}$$

(a) Find the electric field $\boldsymbol{E}_{rad}(\boldsymbol{r}, t)$ in the radiation zone (r very large compared to the system's characteristic diameter and the speed of light times its characteristic time).

(b) Calculate the radiated power delivered $\frac{dP_{rad}}{d\Omega}$ per unit solid angle averaged over one cycle of the system. Note that this problem is equivalent to the case of Fraunhofer (far-field) diffraction of plane waves by two small slits a nonzero distance apart.

Problem 4

A current distribution consists of N identical sources. The kth source is identical to the first source except for a rigid translation by an amount $\boldsymbol{R}_k$ ($k = 1, 2, \ldots, N$). The sources oscillate at the same frequency ω but have different phases δ_k. That is,

$$\boldsymbol{J}_k \propto e^{-i(\omega t + \delta_k)} \tag{9}$$

(a) Show that the angular distribution of radiated power can be written as the product of two factors:one is the angular distribution for $N = 1$; the other depends on $\boldsymbol{R}_k$

(b) The planes of two square loops (each with side length a) are centered on (and lie perpendicular to) the z-axis at $z = \pm a/2$. The loop edges are parallel to the x and y coordinate axes. Find the angular distribution of power, $\frac{dP}{d\Omega}$, in the $x - z$ plane if the current at all points in both loops is $I \cos \omega t$. Make a polar plot of the angular distribution for $\omega a/c = 2\pi$ and $\omega a/c \ll 1$. Identify the multipole character of the radiation in the latter case.

(c) Repeat part (b) when the current in the upper loop is $I \cos \omega t$ and the current in the lower loop is $-I \cos \omega t$.

Problem 5

Two identical point charges q are fixed to the ends of a rod of length 2ℓ which rotates with a constant angular velocity $\frac{1}{2}\omega$ in the $x-y$ plane about an axis perpendicular to the rod and through its center.

(a) Calculate the electric dipole moment and $\boldsymbol{p}(t)$. Is there electric dipole radiation?

(b) Calculate the magnetic dipole moment and $\boldsymbol{m}(t)$. Is there magnetic dipole radiation?

(c) Show that the electric quadrupole moment is

$$\boldsymbol{Q}(t) = \frac{1}{2}q\ell^2 \begin{pmatrix} 1+\cos\omega t & \sin\omega t & 0 \\ \sin\omega t & 1-\cos\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{10}$$

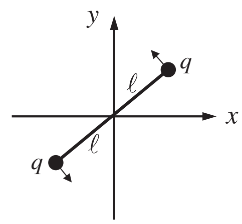

Figure 1

(d) Show that the time-averaged angular distribution of radiated power is

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\mu_0}{4\pi}\frac{q^2\omega^6\ell^4}{32\pi c^3}(1-\cos^4\theta) \tag{11}$$

where θ is the polar angle measured from the z-axis.