Phys 232 Problem Set 3

Released: 2/24/2020
Due: 3/11/2020

References: Jackson Chapters 9 and 11, Zangwill Chapters 15, 20, and 22

Problem 1

Recall that the linear momentum density is given by

$$\boldsymbol{g} = \frac{\boldsymbol{S}}{c^2} = \epsilon_0(\boldsymbol{E} \times \boldsymbol{B}) \tag{1}$$

The angular momentum density is then $\boldsymbol{r} \times \boldsymbol{g}$ which leads to the following expression for the electromagnetic angular momentum $\boldsymbol{L}_{EM}$

$$\boldsymbol{L}_{EM} = \epsilon_0 \int d^3r(\boldsymbol{r} \times \boldsymbol{E} \times \boldsymbol{B}) \tag{2}$$

(a) Show that a classical oscillating electric dipole $\boldsymbol{p}$ with fields given by

$$\boldsymbol{B} = \frac{ck^2}{4\pi}(\hat{\boldsymbol{n}} \times \boldsymbol{p})\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right) \tag{3}$$

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\left[k^2(\hat{\boldsymbol{n}} \times \boldsymbol{p}) \times \hat{\boldsymbol{n}}\frac{e^{ikr}}{r} + (3\hat{\boldsymbol{n}}(\hat{\boldsymbol{n}} \cdot \boldsymbol{p}) - \boldsymbol{p})\left(\frac{1}{r^3} - \frac{ik}{r^2}\right)e^{ikr}\right] \tag{4}$$

radiates electromagnetic angular momentum to infinity at the rate

$$\frac{d\boldsymbol{L}_{EM}}{dt} = \frac{k^3}{12\pi\epsilon_0}\Im[\boldsymbol{p}^* \times \boldsymbol{p}] \tag{5}$$

For more on electromagnetic angular momentum, see Zangwill Chapter 15 section 6

(b) What is the ratio of angular momentum radiated to the energy radiated? Interpret.

(c) For a charge e rotating in the $x - y$ plane at radius a and angular speed ω, show that there is only a z component of radiated angular momentum with magnitude $\frac{d(L_{EM})_z}{dt} = \frac{e^2k^3a^2}{6\pi\epsilon_0}$. What about a charge oscillating along the z axis?

(d) What are the results corresponding to parts a and b for magnetic dipole radiation? *Hint*: The electromagnetic angular momentum density comes from more than the transverse (radiation zone) components of the fields.

Problem 2

An infinitely long straight wire of negligible cross-sectional area is at rest and has a uniform linear charge density q_0 in the intertial frame K'. The frame K' (and the wire) move with a velocity $\boldsymbol{v}$ parallel to the directiomn of the wire with respect to the laboratory frame K.

(a) Write down the electric and magnetic fields in cylindrical coordinates in the rest frame of the wire. Using the Lorentz transformation properties of the fields, find the components of the electric and magnetic fields in the laboratory.

(b) What are the charge and current densities associated with the wire in its rest frame? In the laboratory?

(c) From the laboratory charge and current densities, calculate directly the electromagnetic fields in the laboratory. Compare with the results of part a.

Problem 3

(a) Express the Lorentz scalars $F^{\alpha\beta}F_{\alpha\beta}$ $\tilde{F}^{\alpha\beta}F_{\alpha\beta}$ and $\tilde{F}^{\alpha\beta}\tilde{F}_{\alpha\beta}$ in terms of $\boldsymbol{E}$ and $\boldsymbol{B}$. Are there any other invariants quadratic in the field strengths $\boldsymbol{E}$ and $\boldsymbol{B}$? Note: $\tilde{F}^{\alpha\beta}$ is the dual field strength tensor and is related to $F^{\alpha\beta}$ via

$$\tilde{F}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta} \tag{6}$$

(b) Is it possible to have an electromagnetic field that appears as a purely electric field in one inertial frame and as a purely magnetic field in some other intertial frame? What are the criteria imposed on $\boldsymbol{E}$ and $\boldsymbol{B}$ such that there is an inertial frame in which there is no electric field?

Problem 4

In a certain reference frame a static uniform electric field E_0 is parallel to the x axis, and a static uniform, magnetic induction $B_0 = 2E_0$ lies in the $x-y$ plane, making an angle θ with the axis. Determine the relative velocity of a reference frame in which the electric and magnetic fields are parallel. What are the fields in that frame for $\theta \ll 1$ and $\theta \to (\pi/2)$?

Problem 5

Show by explicit calculation that the formulae for the charge and current densities of a collection of point charges,

$$\rho(\boldsymbol{r}, t) = \sum_{k=1}^{N} q_k \delta(\boldsymbol{r} - \boldsymbol{r}_k(t)) \tag{7}$$

$$\boldsymbol{J}(\boldsymbol{r}, t) = \sum_{k=1}^{N} q_k \boldsymbol{v}_k(t) \delta(\boldsymbol{r} - \boldsymbol{r}_k(t)) \tag{8}$$

have exactly the same form when we boost from frame K to frame K' by a velocity $\boldsymbol{v}_0$.
Hint: Show first that $\delta(\boldsymbol{r}' - \boldsymbol{r}'_k(t')) = \gamma(1 - \boldsymbol{v}_0 \cdot \boldsymbol{v}_k/c^2)\delta(\boldsymbol{r} - \boldsymbol{r}_k(t))$ by evaluating the Jacobian determinant in the volume element transformation $d^3R = |\boldsymbol{J}(\boldsymbol{R}, \boldsymbol{r}')|d^3r'$ where $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}_k(t)$.