Phys 232 Problem Set 4

Released: 3/09/2020
Due: 3/30/2020

References: Jackson chapters 6 and 12, Zangwill chapters 23 and 24

Problem 1

Construct a covariant expression for the rate at which a moving charged particle loses total energy-momentum $P^\mu = (U_{rad}/c, \boldsymbol{P}_{rad})$, i.e. $dP^\mu/d\tau$. Evaluate your expression in an arbitrary inertial frame as a check.

Problem 2

The free-field Lagrangian density,

$$\mathcal{L}_P = \frac{1}{2}\epsilon_0 \left[\left(\nabla\varphi + \frac{\partial \boldsymbol{A}}{\partial t} \right)^2 - c^2 (\nabla \times \boldsymbol{A})^2 \right] - \frac{1}{2\mu_0 \ell^2} \left[\boldsymbol{A}^2 - (\varphi/c)^2 \right] \tag{1}$$

with $\ell = \hbar/mc$ was introduced by Alexandre Proca in 1936 as an alternative to Dirac's theory of the positron. Today, it serves as a model for electrodynamics with a photon with mass m when matter-field coupling is added to get the total Lagrangian density, $\mathcal{L} = \boldsymbol{J} \cdot \boldsymbol{A} - \rho\varphi + \mathcal{L}_P$

(a) Find the effect of the Proca mass term on the Maxwell equations.

(b) Show that the Proca model violates gauge invariance because a particular choice of gauge must be made to guarantee conservation of charge.

(c) Find the scalar potential for a static point charge q in the Proca model.

Problem 3

A model for an electrodynamics which respects gauge invariance but violates Lorentz invariance supplements the usual Maxwell Lagrangian with terms drawn from a four-vector $d^\mu = (d_0, \boldsymbol{d})$:

$$L_{CS} = \int d^3 r \left[\rho\varphi - \boldsymbol{J} \cdot \boldsymbol{A} + \frac{1}{2} \{ \epsilon_0 (\boldsymbol{E}^2 - c^2 \boldsymbol{B}^2) - \varphi(\boldsymbol{d} \cdot \boldsymbol{B}/c) + \boldsymbol{d} \cdot (\boldsymbol{A} \times \boldsymbol{E}/c) + d_0 \boldsymbol{A} \cdot \boldsymbol{B} \} \right] \tag{2}$$

(a) Find the restrictions that must be imposed on d_μ to ensure that a gauge transformation does not alter the dynamics.

(b) Assume that d^μ is a *constant* four-vector. Is this consistent with your answer to part (a)? Find the Chern-Simons Maxwell equations (Euler Lagrange equation for L_{CS}) which replace the usual Maxwell equations. Confirm that the theory is gauge invariant but does not respect Lorentz invariance.

Problem 4

(a) For a particle possessing both electric and magnetic charges, show that the generalization of the Lorentz force in vacuum is

$$\boldsymbol{F} = q_e (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) + q_m \left(\boldsymbol{B} - \frac{\boldsymbol{v}}{c^2} \times \boldsymbol{E} \right) \tag{3}$$

(b) Show that this expression for the force is invariant under a duality transformation of both the fields

$$\boldsymbol{E} = \boldsymbol{E}' \cos\xi + c\boldsymbol{B}' \sin\xi \tag{4}$$

$$\boldsymbol{B} = -\frac{1}{c}\boldsymbol{E}' \sin\xi + \boldsymbol{B}' \cos\xi \tag{5}$$

and charges

$$\rho_e = \rho_e' \cos\xi + \frac{1}{c}\rho_m' \sin\xi, \quad \boldsymbol{J}_e = \boldsymbol{J}_e' \cos\xi + \frac{1}{c}\boldsymbol{J}_m' \sin\xi \tag{6}$$

$$\rho_m = -c\rho_e' \sin\xi + \rho_m' \cos\xi, \quad \boldsymbol{J}_m = -c\boldsymbol{J}_e' \sin\xi + \boldsymbol{J}_m' \cos\xi \tag{7}$$

(c) Show that the Dirac quantization condition

$$n = \frac{eg}{2\pi\hbar\epsilon_0 c^2}, n \in \mathbb{Z} \tag{8}$$

is generalized for two particles possessing electric and magnetic charges e_1, g_1 and e_2, g_2 respectively to

$$\frac{e_1 g_2 - e_2 g_1}{\hbar\epsilon_0 c^2} = 2\pi n \tag{9}$$

and that the relation is invariant under a duality transformation of the charges.