Phys 232 Problem Set 6

Released: 4/15/2020
Due: 4/29/2020

References: Zangwill Chapters 18, 19, J. Goldstone and R.L. Jaffe, *Physical Review B* **45**, 14100 (1992).

Problem 1

An electromagnetic wave $\boldsymbol{E} = \delta\boldsymbol{E}\exp(-i\omega t)$ can induce a net magnetization in a metal. To see this, let the density and velocity of the electrons at a typical point be $n = \bar{n} + \delta n \exp(-i\omega t)$ and $\boldsymbol{v} = \bar{\boldsymbol{v}} + \delta\boldsymbol{v}\exp(-i\omega t)$, where $\bar{n}$ is the mean density of the electrons and $\bar{\boldsymbol{v}} = 0$ is the mean velocity of the electrons. The current density $\boldsymbol{j} = -en\boldsymbol{v}$ has two time-dependent pieces, which is $\delta\boldsymbol{j} = -e\bar{n}\delta\boldsymbol{v} = \sigma\delta\boldsymbol{E}$, where $\sigma = i\bar{n}e^2/m\omega$ is the collisionless Drude conductivity.

(a) Show that the time-averaged current density is $\langle \boldsymbol{j} \rangle = -\frac{1}{2}\Re\{e\delta n\delta\boldsymbol{v}^*\}$

(b) Evaluate δn to first order in $\delta\boldsymbol{v}$ (using the continuity equation) and show that a piece of $\langle \boldsymbol{j} \rangle$ has the form $\nabla \times \boldsymbol{M}$ where (the plasma frequency is defined by $\omega_p^2 = ne^2/m\epsilon_0$)

$$\boldsymbol{M} = \frac{i\epsilon_0 e\omega_p^2}{4m\omega^3}(\delta\boldsymbol{E} \times \delta\boldsymbol{E}^*) \tag{1}$$

(c) Evaluate $\boldsymbol{M}$ when $\delta\boldsymbol{E}$ is linearly polarized. Repeat for circular polarization.

Problem 2

Drude's conductivity formula fails when the frequency ω is low and the mean time τ between electron collisions is large. If $\bar{v}$ is a characteristic electron speed, one says that the normal skin effect becomes anomalous when the mean distance between collisions $\ell = \bar{v}\tau$ exceeds the skin depth $\delta(\omega)$. To study this regime, we first write the rate of change of an ohmic current density $\boldsymbol{j}(t)$ as the sum of a field-driven acceleration term $d\boldsymbol{j}/dt_{\text{acc}} = (\sigma_0/\tau)\boldsymbol{E}$ and a collisional deceleration term $d\boldsymbol{j}/dt_{\text{coll}} = \boldsymbol{j}/\tau$. This reproduces Ohm's law in the steady state $d\boldsymbol{j}/dt = 0$ because

$$\frac{d\boldsymbol{j}}{dt} = \frac{\sigma_0}{\tau}\boldsymbol{E} - \frac{\boldsymbol{j}}{\tau} \tag{2}$$

(a) Approximate $d\boldsymbol{j}/dt$ by $\partial\boldsymbol{j}/\partial t$ and combine the foregoing with the Maxwell equations neglecting the displacement current) to get a partial differential for $\boldsymbol{B}(\boldsymbol{r}, t)$ that has only first-order time derivatives:

$$\nabla^2\left[\boldsymbol{B} + \tau\frac{\partial\boldsymbol{B}}{\partial t}\right] = \mu_0\sigma_0\frac{\partial\boldsymbol{B}}{\partial t} \tag{3}$$

Let $\boldsymbol{B}(z, t) = \boldsymbol{B}_0 e^{i(kz-\omega t)}$ and confirm that Drude's frequency-dependent conductivity emerges from your dispersion relation $k(\omega)$.

(b) Drude's conductivity formula overestimates the effect of collisions when $\ell \gg \delta$. A phenomenological way to correct this exploits the convective derivative to write

$$\frac{d\boldsymbol{j}}{dt} = \frac{\partial\boldsymbol{j}}{\partial t} - \bar{v}\frac{\partial\boldsymbol{j}}{\partial z} \tag{4}$$

Derive a cubic equation which determines the new dispersion relation. Find $k(\omega)$ explicitly in the extreme anomalous limit (where the gradient term dominates) and show that

$$\boldsymbol{B}(z,t) = \boldsymbol{B}_0 \exp\left((i-\sqrt{3})z/\delta^*(\omega)\right) e^{-i\omega t} \tag{5}$$

The anomalous skin depth $\delta^*(\omega) = 2(\Lambda^2 \bar{v}/\omega)^{1/3}$ found here describes experiments well in this regime. The constant $\Lambda^2 = m/\mu_0 n e^2$.

(c) Show that $\ell \ll \delta = \sqrt{2/\mu_0 \omega \sigma_0}$ is the condition to neglect the non-local gradient term.

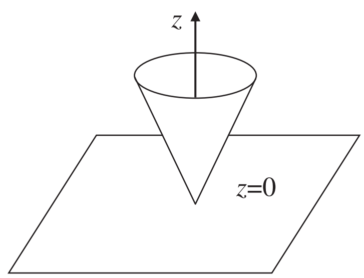
Problem 3

Consider time-harmonic solutions to the Maxwell equations in vacuum where the fields are independent of the azimuthal angle ϕ. TEM solutions of this type also have no radial component to the fields: $E_r = B_r = 0$.

(a) Show that the conditions stated above decouple the Maxwell curl equations into two subsets, each of which describes a different type of TEM wave.

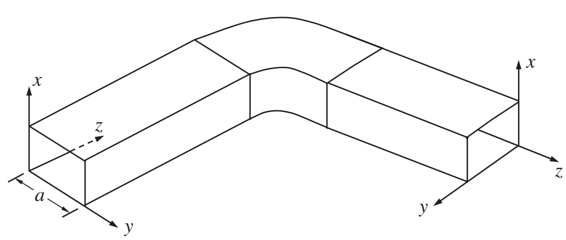
(b) Begin with the Maxwell divergence equations and find general solutions for $E(r,\theta,t)$ and $B(r,\theta,t)$ for each of the two TEM wave types.

(c) The figure below shows the apex of an infinite, solid conducting cone touching the conducting half-space $z < 0$. Explain why this structure can be used to guide one of the TEM wave types found above but not the other.



Problem 4

A rectangular waveguide with a constant cross section and perfectly conducting walls contains a curved section as sketched below. Also indicated is a local Cartesian coordinate system where the z-axis and y-axis remain tangent and normal to the walls, respectively.



(a) The scalar function Φ satisfies $[\nabla_\perp^2 + \omega^2/c^2]\Phi(y,z) = 0$, where $\nabla_\perp^2 = \partial^2/\partial y^2 + \partial^2/\partial z^2$. how that the four vacuum Maxwell equations and conducting wall-boundary conditions are satisfied by time-harmonic transverse electric (TE) modes of the form

$$\boldsymbol{E} = \hat{\boldsymbol{x}} i \frac{\omega}{c} \Phi \tag{6}$$

$$c\boldsymbol{B} = -\hat{\boldsymbol{x}} \times \nabla \Phi \tag{7}$$

(b) Suppose that the curvature $\kappa(z)$ of the side wall at any point on the guide satisfies $\kappa a \ll 1$ so the Laplacian operator in the local coordinate Cartesian coordinate system is well approximated by

$$\nabla^2 = \partial^2/\partial y^2 + \partial^2/\partial z^2 + \frac{1}{2}\kappa^2(z) \tag{8}$$

Separate variables in the Helmholtz equation and show that propagating modes exist in the straight portion of the guide (at least) when $\omega > \pi c/a$.

(c) Show that at least one mode exists in the curved part of the guide for $\omega < \pi c/a$. Describe the spatial characteristics of this solution. Hint: Make an analogy with the one-dimensional, time-independent Schrödinger equation.