Phys 232 Problem Set 7 / Final

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Please let us know if you do not have access to Zangwill.

Problem 1

Consider a one-dimensional photonic crystal made of two alternating dielectric layers with the dielectric constants ϵ_1 and ϵ_2 and different thicknesses d_1 and d_2. Derive the characteristic equation for TE and TM modes. Comment on how the dispersion curve $k_x(\omega)$ depends on d_2/d_1. Generalize the analysis in Section 11.1.1 of Novotny and Hecht, available on the class web site (in the Additional Reading folder).

Problem 2

A low-frequency, plane electromagnetic wave Rayleigh scatters from a sphere with radius a and conductivity σ. Assume that the skin depth $\delta \gg a$.

(a) Find the electric dipole moment, $\boldsymbol{p}$ induced in the sphere by the incident wave. Argue that in the limit $\delta \gg a$, we can just treat the sphere as a dielectric sphere with the the frequency-dependent permittivity $\hat{\epsilon}(\omega) = \epsilon + i\sigma/\omega$ (Zangwill Section 17.6). So the problem reduces to finding the dipole moment of a sphere in the presence of a uniform electric field. One can identify $\boldsymbol{p}$ from a multipole expansion of the potential with the appropriate matching conditions inside and outside of the sphere (see Zangwill Section 6.5).

(b) Calculate the scattering cross section of the sphere.

Problem 3

Consider the lowest order approximation to the scattering by a large uniform dielectric sphere of radius a, much larger than the wavelength ($ka \gg 1$), whose dielectric constant ϵ_r only differs slightly from unity. Show that the differential cross section is sharply peaked in the forward direction and the total scattering cross section is approximately

$$\sigma \approx \frac{\pi}{2}(ka)^2|\epsilon_r - 1|^2 a^2 \tag{1}$$

with a k^2 rather than the k^4 dependence on frequency present for $ka \ll 1$. Start by considering the Born approximation to the differential cross section for scattering from a uniform and lossless dielectric sphere in example 21.3 of Zangwill.

In the scattering of light by a gas very near the critical point, the scattered light is observed to be "whiter" (i.e. its spectrum is less predominately peaked toward the blue) than far from the critical point. From this problem, we see that this can be understood by the fact that the volumes of the density fluctuations become large enough that Rayleigh's law (i.e. $ka \ll 1$ considered in class) fails to hold.

Problem 4

A monochromatic plane wave scatters from a Lorentz atom where a bound electron obeys the classical equation of motion $\ddot{\boldsymbol{r}} + \gamma\dot{\boldsymbol{r}} + \omega_0^2\boldsymbol{r} = 0$. Assume that the electron displacement and damping are both very small. $r_e = e^2/(4\pi\epsilon_0 mc^2)$ is the classical electron radius.

1. Calculate the differential scattering cross section $d\sigma_{\text{scatt}}/d\Omega$
2. Read Section 21.7 until 21.7.1 in Zangwill on the definition of the total and absorption cross section. Calculate absorption cross section $\sigma_{\text{abs}}(\omega)$
3. Show that the integrated absorption cross section is independent of the damping constant:

$$\int_0^\infty d\omega \sigma_{\text{abs}}(\omega) = 2\pi^2 r_e c \tag{2}$$

Problem 5

A monochromatic plane wave polarized along $\hat{\boldsymbol{y}}$ is normally incident from $z < 0$ onto a two-dimensional conducting scatterer confined to the $z = 0$ plane. Use Kirchoff's approximation

$$\hat{\boldsymbol{z}} \times \boldsymbol{E} \approx 0 \text{ on the screen} \tag{3}$$
$$\hat{\boldsymbol{z}} \times \boldsymbol{E} \approx \hat{\boldsymbol{z}} \times \boldsymbol{E}_{\text{inc}} \text{ in the aperture} \tag{4}$$

but do not use the Fraunhofer approximation (i.e. the distance from the point of observation may not be large relative to the size of the aperture). The needed formulae are in Zangwill Section 21.8.2

(a) Let the scatterer be a conducting disk of radius a. Find $E_{\text{disk}}(0, 0, z > 0)$.

(b) Let the scatterer be an infinite conducting sheet with a circular aperture of radius a centered on the z-axis. Find $E_{\text{aperture}}(0, 0, z > 0)$.

(c) Confirm that $E_{\text{aperture}}(0, 0, z > 0) = E_{\text{inc}} - E_{\text{disk}}(0, 0, z > 0)$.